

# Finsler and Lagrange Geometries in Einstein and String Gravity

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## Abstract

We review the current status of Finsler–Lagrange geometry and generalizations. The goal is to aid non-experts on Finsler spaces, but physicists and geometers skilled in general relativity and particle theories, to understand the crucial importance of such geometric methods for applications in modern physics. We also would like to orient mathematicians working in generalized Finsler and Kähler geometry and geometric mechanics how they could perform their results in order to be accepted by the community of "orthodox" physicists.

Although the bulk of former models of Finsler–Lagrange spaces where elaborated on tangent bundles, the surprising result advocated in our works is that such locally anisotropic structures can be modelled equivalently on Riemann–Cartan spaces, even as exact solutions in Einstein and/or string gravity, if nonholonomic distributions and moving frames of references are introduced into consideration.

We also propose a canonical scheme when geometrical objects on a (pseudo) Riemannian space are nonholonomically deformed into generalized Lagrange, or Finsler, configurations on the same manifold. Such canonical transforms are defined by the coefficients of a prime metric and generate target spaces as Lagrange structures, their models of almost Hermitian/ Kähler, or nonholonomic Riemann spaces.

Finally, we consider some classes of exact solutions in string and Einstein gravity modelling Lagrange–Finsler structures with solitonic pp-waves and speculate on their physical meaning.

**Keywords:** Nonholonomic manifolds, Einstein spaces, string gravity, Finsler and Lagrange geometry, nonlinear connections, exact solutions, Riemann–Cartan spaces.

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## 1 Introduction

The main purpose of this survey is to present an introduction to Finsler–Lagrange geometry and the anholonomic frame method in general relativity and gravitation. We review and discuss possible applications in modern physics and provide alternative constructions in the language of the geometry of nonholonomic Riemannian manifolds (enabled with nonintegrable

distributions and preferred frame structures). It will be emphasized the approach when Finsler like structures are modelled in general relativity and gravity theories with metric compatible connections and, in general, non-trivial torsion.

Usually, gravity and string theory physicists may remember that Finsler geometry is a quite "sophisticate" spacetime generalization when Riemannian metrics  $g_{ij}(x^k)$  are extended to Finsler metrics  $g_{ij}(x^k, y^l)$  depending both on local coordinates  $x^k$  on a manifold  $M$  and "velocities"  $y^l$  on its tangent bundle  $TM$ .<sup>1</sup> Perhaps, they will say additionally that in order to describe local anisotropies depending only on directions given by vectors  $y^l$ , the Finsler metrics should be defined in the form  $g_{ij} \sim \frac{\partial F^2}{\partial y^i \partial y^j}$ , where  $F(x^k, \zeta y^l) = |\zeta| F(x^k, y^l)$ , for any real  $\zeta \neq 0$ , is a fundamental Finsler metric function. A number of authors analyzing possible locally anisotropic physical effects omit a rigorous study of nonlinear connections and do not reflect on the problem of compatibility of metric and linear connection structures. If a Riemannian geometry is completely stated by its metric, various models of Finsler spaces and generalizations are defined by three independent geometric objects (metric and linear and nonlinear connections) which in certain canonical cases are induced by a fundamental Finsler function  $F(x, y)$ . For models with different metric compatibility, or non-compatibility, conditions, this is a point of additional geometric and physical considerations, new terminology and mathematical conventions. Finally, a lot of physicists and mathematicians have concluded that such geometries with generic local anisotropy are characterized by various types of connections, torsions and curvatures which do not seem to have physical meaning in modern particle theories but (may be?) certain Finsler like analogs of mechanical systems and continuous media can be constructed.

There were published a few rigorous studies on perspectives of Finsler like geometries in standard theories of gravity and particle physics (see, for instance, Refs. [12, 85]) but they do not analyze any physical effects of the nonlinear connection and adapted linear connection structures and the possibility to model Finsler like spaces as exact solutions in Einstein and string gravity [77]). The results of such works, on Finsler models with violations of local Lorentz symmetry and nonmetricity fields, can be summarized in a very pessimistic form: both fundamental theoretic consequences and experimental data restrict substantially the importance for modern physics of locally anisotropic geometries elaborated on (co) tangent bundles,<sup>2</sup> see In-

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<sup>1</sup>we emphasize that Finsler geometries can be alternatively modelled if  $y^l$  are considered as certain nonholonomic, i. e. constrained, coordinates on a general manifold  $\mathbf{V}$ , not only as "velocities" or "momenta", see further constructions in this work

<sup>2</sup>In result of such opinions, the Editors and referees of some top physical journals almost stopped to accept for publication manuscripts on Finsler gravity models. If other journals were more tolerant with such theoretical works, they were considered to be related to certain alternative classes of theories or to some mathematical physics problems

roduction to monograph [77] and article [64] and reference therein for more detailed reviews and discussions.

Why we should give a special attention to Finsler geometry and methods and apply them in modern physics ? We list here a set of contr-arguments and discuss the main sources of "anti-Finsler" skepticism which (we hope) will explain and re-move the existing unfair situation when spaces with generic local anisotropy are not considered in standard theories of physics:

1. One should be emphasized that in the bulk the criticism on locally anisotropic geometries and applications in standard physics was motivated only for special classes of models on tangent bundles, with violation of local Lorentz symmetry (even such works became very important in modern physics, for instance, in relation to brane gravity [16] and quantum theories [28]) and nonmetricity fields. Not all theories with generalized Finsler metrics and connections were elaborated in this form (on alternative approaches, see next points) and in many cases, like [12, 85], the analysis of physical consequences was performed not following the nonlinear connection geometric formalism and a tensor calculus adapted to nonholonomic structures which is crucial in Finsler geometry and generalizations.
2. More recently, a group of mathematicians [8, 54] developed intensively some directions on Finsler geometry and applications following the Chern's linear connection formalism proposed in 1948 (this connection is with vanishing torsion but noncompatible with the metric structure). For non-experts in geometry and physics, the works of this group, and other authors working with generalized local Lorentz symmetries, created a false opinion that Finsler geometry can be elaborated only on tangent bundles and that the Chern connection is the "best" Finsler generalization of the Levi Civita connection. A number of very important constructions with the so-called metric compatible Cartan connection, or other canonical connections, were moved on the second plan and forgotten. One should be emphasized that the geometric constructions with the well known Chern or Berwald connections can not be related to standard theories of physics because they contain nonmetricity fields. The issue of nonmetricity was studied in details in a number of works on metric-affine gravity, see review [22] and Chapter I in the collection of works [77], the last one containing a series of papers on generalized Finsler-affine spaces. Such results are not widely accepted by physicists because of absence of experimental evidences and theoretical complexity of geometric constructions. Here we note that it is a quite sophisticate task to elaborate spinor ver-

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with speculations on geometric models and "nonstandard" physics, mechanics and some applications to biology, sociology or seismology etc

sions, supersymmetric and noncommutative generalizations of Finsler like geometries if we work with metric noncompatible connections.

3. A non-expert in special directions of differential geometry and geometric mechanics, may not know that beginning E. Cartan (1935) [15] various models of Finsler geometry were developed alternatively by using metric compatible connections which resulted in generalizations to the geometry of Lagrange and Hamilton mechanics and their higher order extensions. Such works and monographs were published by prominent schools and authors on Finsler geometry and generalizations from Romania and Japan [37, 38, 34, 35, 39, 36, 25, 26, 24, 57, 84, 33, 41, 42, 44, 45, 10, 11] following approaches quite different from the geometry of symplectic mechanics and generalizations [30, 31, 32, 29]. As a matter of principle, all geometric constructions with the Chern and/or symplectic connections can be redefined equivalently for metric compatible geometries, but the philosophy, aims, mathematical formalism and physical consequences are very different for different approaches and the particle physics researches usually are not familiar with such results.
4. It should be noted that for a number of scientists working in Western Countries there are less known the results on the geometry of nonholonomic manifolds published in a series of monographs and articles by G. Vranceanu (1926), Z. Horak (1927) and others [80, 81, 82, 23], see historical remarks and bibliography in Refs. [11, 77]. The importance for modern physics of such works follows from the idea and explicit proofs (in quite sophisticated component forms) that various types of locally anisotropic geometries and physical interactions can be modelled on usual Riemannian manifolds by considering nonholonomic distributions and holonomic fibrations enabled with certain classes of special connections.
5. In our works (see, for instance, reviews and monographs [58, 59, 60, 61, 62, 76, 78, 79, 64, 77], and references therein), we re-oriented the research on Finsler spaces and generalizations in some directions connected to standard models of physics and gauge, supersymmetric and noncommutative extensions of gravity. Our basic idea was that both the Riemann–Cartan and generalized Finsler–Lagrange geometries can be modelled in a unified manner by corresponding geometric structures on nonholonomic manifolds. It was emphasized, that prescribing a preferred nonholonomic frame structure (equivalently, a nonintegrable distribution with associated nonlinear connection) on a manifold, or on a vector bundle, it is possible to work equivalently both with the Levi Civita and the so-called canonical distinguished connection. We provided a number of examples when Finsler like structures and

geometries can be modelled as exact solutions in Einstein and string gravity and proved that certain geometric methods are very important, for instance, in constructing new classes of exact solutions.

This review work has also pedagogical scopes. We attempt to cover key aspects and open issues of generalized Finsler–Lagrange geometry related to a consistent incorporation of nonlinear connection formalism and moving/deformation frame methods into the Einstein and string gravity and analogous models of gravity, see also Refs. [64, 77, 38, 10, 53] for general reviews, written in the same spirit as the present one but in a more comprehensive, or inversely, with more special purposes forms. While the article is essentially self-contained, the emphasis is on communicating the underlying ideas and methods and the significance of results rather than on presenting systematic derivations and detailed proofs (these can be found in the listed literature).

The subject of Finsler geometry and applications can be approached in different ways. We choose one of which is deeply rooted in the well established gravity physics and also has sufficient mathematical precision to ensure that a physicist familiar with standard textbooks and monographs on gravity [21, 40, 83, 56] and string theory [18, 51, 55] will be able without much efforts to understand recent results and methods of the geometry of nonholonomic manifolds and generalized Finsler–Lagrange spaces.

We shall use the terms "standard" and "nonstandard" models in geometry and physics. In connection to Finsler geometry, we shall consider a model to be a standard one if it contains locally anisotropic structures defined by certain nonholonomic distributions and adapted frames of reference on a (pseudo) Riemannian or Riemann–Cartan space (for instance, in general relativity, Kaluza–Klein theories and low energy string gravity models). Such constructions preserve, in general, the local Lorentz symmetry and they are performed with metric compatible connections. The term "nonstandard" will be used for those approaches which are related to metric non-compatible connections and/or local Lorentz violations in Finsler spacetimes and generalizations. Sure, any standard or nonstandard model is rigorously formulated following certain purposes in modern geometry and physics, geometric mechanics, biophysics, locally anisotropic thermodynamics and stochastic and kinetic processes and classical or quantum gravity theories. Perhaps, it will be the case to distinguish the class of "almost standard" physical models with locally anisotropic interactions when certain geometric objects from a (pseudo) Riemannian or Riemann–Cartan manifolds are lifted on a (co) tangent or vector bundles and/or their supersymmetric, non-commutative, Lie algebroid, Clifford space, quantum group ... generalizations. There are possible various effects with "nonstandard" corrections, for instance, violations of the local Lorentz symmetry by quantum effects but in some classical or quantum limits such theories are constrained to correspond to certain standard ones.

This contribution is organized as follows:

In section 2, we outline an unified approach to the geometry of nonholonomic distributions on Riemann manifolds and Finsler–Lagrange spaces. The basic concepts on nonholonomic manifolds and associated nonlinear connection structures are explained and the possibility of equivalent (non) holonomic formulations of gravity theories is analyzed.

Section 3 is devoted to nonholonomic deformations of manifolds and vector bundles. There are reviewed the basic constructions in the geometry of (generalized) Lagrange and Finsler spaces. A general ansatz for constructing exact solutions, with effective Lagrange and Finsler structures, in Einstein and string gravity, is analyzed.

In section 4, the Finsler–Lagrange geometry is formulated as a variant of almost Hermitian and/or Kähler geometry. We show how the Einstein gravity can be equivalently reformulated in terms of almost Hermitian geometry with preferred frame structure.

Section 5 is focused on explicit examples of exact solutions in Einstein and string gravity when (generalized) Finsler–Lagrange structures are modelled on (pseudo) Riemannian and Riemann–Cartan spaces. We analyze some classes of Einstein metrics which can be deformed into new exact solutions characterized additionally by Lagrange–Finsler configurations. For string gravity, there are constructed explicit examples of locally anisotropic configurations describing gravitational solitonic pp–waves and their effective Lagrange spaces. We also analyze some exact solutions for Finsler–solitonic pp–waves on Schwarzschild spaces.

Conclusions and further perspectives of Finsler geometry and new geometric methods for modern gravity theories are considered in section 6.

Finally, we should note that our list of references is minimalist, trying to concentrate on reviews and monographs rather than on original articles. More complete reference lists are presented in the books [77, 62, 76, 38, 39]. Various guides for learning, both for experts and beginners on geometric methods and further applications in standard and nonstandard physics, with references, are contained in [77, 38, 39, 10, 53].

## 2 Nonholonomic Einstein Gravity and Finsler–Lagrange Spaces

In this section we present in a unified form the Riemann–Cartan and Finsler–Lagrange geometry. The reader is supposed to be familiar with well-known geometrical approaches to gravity theories [21, 40, 83, 56] but may not know the basic concepts on Finsler geometry and nonholonomic manifolds. The constructions for locally anisotropic spaces will be derived by special parametrizations of the frame, metric and connection structures on usual manifolds, or vector bundle spaces, as we proved in details in Refs. [77, 64].

## 2.1 Metric-affine, Riemann-Cartan and Einstein manifolds

Let  $V$  be a necessary smooth class manifold of dimension  $\dim V = n + m$ , when  $n \geq 2$  and  $m \geq 1$ , enabled with **metric**,  $g = g_{\alpha\beta}e^\alpha \otimes e^\beta$ , and **linear connection**,  $D = \{\Gamma_{\beta\gamma}^\alpha\}$ , structures. The coefficients of  $g$  and  $D$  can be computed with respect to any local **frame**,  $e_\alpha$ , and **co-frame**,  $e^\beta$ , bases, for which  $e_\alpha \rfloor e^\beta = \delta_\alpha^\beta$ , where  $\rfloor$  denotes the interior (scalar) product defined by  $g$  and  $\delta_\alpha^\beta$  is the Kronecker symbol. A local system of coordinates on  $V$  is denoted  $u^\alpha = (x^i, y^a)$ , or (in brief)  $u = (x, y)$ , where indices run correspondingly the values:  $i, j, k, \dots = 1, 2, \dots, n$  and  $a, b, c, \dots = n+1, n+2, \dots, n+m$  for any splitting  $\alpha = (i, a), \beta = (j, b), \dots$ . We shall also use primed, underlined, or other type indices: for instance,  $e_{\alpha'} = (e_{i'}, e_{a'})$  and  $e^{\beta'} = (e^{j'}, e^{b'})$ , for a different sets of local (co) bases, or  $\underline{e}_\alpha = e_{\underline{\alpha}} = \partial_{\underline{\alpha}} = \partial/\partial u^{\underline{\alpha}}$ ,  $\underline{e}_i = e_{\underline{i}} = \partial_{\underline{i}} = \partial/\partial x^{\underline{i}}$  and  $\underline{e}_a = e_{\underline{a}} = \partial_{\underline{a}} = \partial/\partial y^{\underline{a}}$  if we want to emphasize that the coefficients of geometric objects (tensors, connections, ...) are defined with respect to a local **coordinate basis**. For simplicity, we shall omit underlining or priming of indices and symbols if that will not result in ambiguities. The Einstein's summation rule on repeating "up-low" indices will be applied if the contrary will not be stated.

**Frame transforms** of a local basis  $e_\alpha$  and its dual basis  $e^\beta$  are parametrized in the form

$$e_\alpha = A_\alpha^{\alpha'}(u)e_{\alpha'} \text{ and } e^\beta = A^\beta_{\beta'}(u)e^{\beta'}, \quad (1)$$

where the matrix  $A^\beta_{\beta'}$  is inverse to  $A_\alpha^{\alpha'}$ . In general, local bases are **non-holonomic** (equivalently, **anholonomic**, or **nonintegrable**) and satisfy certain anholonomy conditions

$$e_\alpha e_\beta - e_\beta e_\alpha = W_{\alpha\beta}^\gamma e_\gamma \quad (2)$$

with nontrivial **anholonomy coefficients**  $W_{\alpha\beta}^\gamma(u)$ . We consider the **holonomic** frames to be defined by  $W_{\alpha\beta}^\gamma = 0$ , which holds, for instance, if we fix a local coordinate basis.

Let us denote the covariant derivative along a vector field  $X = X^\alpha e_\alpha$  as  $D_X = X \rfloor D$ . One defines three fundamental geometric objects on manifold  $V$ : **nonmetricity** field,

$$\mathcal{Q}_X \doteq D_X g, \quad (3)$$

**torsion**,

$$\mathcal{T}(X, Y) \doteq D_X Y - D_Y X - [X, Y], \quad (4)$$

and **curvature**,

$$\mathcal{R}(X, Y)Z \doteq D_X D_Y Z - D_Y D_X Z - D_{[X, Y]}Z, \quad (5)$$

where the symbol " $\doteq$ " states "by definition" and  $[X, Y] \doteq XY - YX$ . With respect to fixed local bases  $e_\alpha$  and  $e^\beta$ , the coefficients  $\mathcal{Q} = \{Q_{\alpha\beta\gamma} =$



$D_\alpha g_{\beta\gamma}\}, \mathcal{T} = \{T^\alpha_{\beta\gamma}\}$  and  $\mathcal{R} = \{R^\alpha_{\beta\gamma\tau}\}$  can be computed by introducing  $X \rightarrow e_\alpha, Y \rightarrow e_\beta, Z \rightarrow e_\gamma$  into respective formulas (3), (4) and (5).

In gravity theories, one uses three others important geometric objects: the **Ricci tensor**,  $Ric(D) = \{R_{\beta\gamma} \doteq R^\alpha_{\beta\gamma\alpha}\}$ , the **scalar curvature**,  $R \doteq g^{\alpha\beta} R_{\alpha\beta}$  ( $g^{\alpha\beta}$  being the inverse matrix to  $g_{\alpha\beta}$ ), and the **Einstein tensor**,  $\mathcal{E} = \{E_{\alpha\beta} \doteq R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R\}$ .

A manifold  ${}^{ma}V$  is a **metric-affine space** if it is provided with arbitrary two independent metric  $g$  and linear connection  $D$  structures and characterized by three nontrivial fundamental geometric objects  $\mathcal{Q}, \mathcal{T}$  and  $\mathcal{R}$ .

If the metricity condition,  $\mathcal{Q} = 0$ , is satisfied for a given couple  $g$  and  $D$ , such a manifold  ${}^{RC}V$  is called a **Riemann-Cartan space** with nontrivial torsion  $\mathcal{T}$  of  $D$ .

A **Riemann space**  ${}^RV$  is provided with a metric structure  $g$  which defines a unique Levi Civita connection  ${}_1D = \nabla$ , which is both metric compatible,  ${}_1\mathcal{Q} = \nabla g = 0$ , and torsionless,  ${}_1\mathcal{T} = 0$ . Such a space is pseudo- (semi-) Riemannian if locally the metric has any mixed signature  $(\pm 1, \pm 1, \dots, \pm 1)$ .<sup>3</sup> In brief, we shall call all such spaces to be Riemannian (with necessary signature) and denote the main geometric objects in the form  ${}_1\mathcal{R} = \{{}_1R^\alpha_{\beta\gamma\tau}\}$ ,  ${}_1Ric({}_1D) = \{{}_1R_{\beta\gamma}\}$ ,  ${}_1R$  and  ${}_1\mathcal{E} = \{{}_1E_{\alpha\beta}\}$ .

The **Einstein gravity theory** is constructed canonically for  $\dim^RV = 4$  and Minkowski signature, for instance,  $(-1, +1, +1, +1)$ . Various generalizations in modern **string and/or gauge gravity** consider Riemann, Riemann-Cartan and metric-affine spaces of higher dimensions.

The **Einstein equations** are postulated in the form

$$\mathcal{E}(D) \doteq Ric(D) - \frac{1}{2} g Sc(D) = \Upsilon, \quad (6)$$

where the source  $\Upsilon$  contains contributions of matter fields and corrections from, for instance, string/brane theories of gravity. In a physical model, the equations (6) have to be completed with equations for the matter fields and torsion (for instance, in the **Einstein-Cartan theory** [22], one considers algebraic equations for the torsion and its source). It should be noted here that because of possible nonholonomic structures on a manifold  $V$  (we shall call such spaces to be locally anisotropic), see next section, the tensor  $Ric(D)$  is not symmetric and  $D[\mathcal{E}(D)] \neq 0$ . This imposes a more sophisticate form of conservation laws on spaces with generic "local anisotropy", see discussion in [77] (a similar situation arises in Lagrange mechanics [30, 31, 32, 29, 38] when nonholonomic constraints modify the definition of conservation laws).

For **general relativity**,  $\dim V = 4$  and  $D = \nabla$ , the field equations can

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<sup>3</sup>mathematicians usually use the term semi-Riemannian but physicists are more familiar with pseudo-Riemannian; we shall apply both terms on convenience

be written in the well-known component form

$${}_i E_{\alpha\beta} = {}_i R_{\beta\gamma} - \frac{1}{2} {}_i R = \Upsilon_{\alpha\beta} \quad (7)$$

when  $\nabla({}_i E_{\alpha\beta}) = \nabla(\Upsilon_{\alpha\beta}) = 0$ . The coefficients in equations (7) are defined with respect to arbitrary nonholonomic frame (1).

## 2.2 Nonholonomic manifolds and adapted frame structures

A **nonholonomic manifold**  $(M, \mathcal{D})$  is a manifold  $M$  of necessary smooth class enabled with a nonholonomic distribution  $\mathcal{D}$ , see details in Refs. [11, 77]. Let us consider a  $(n + m)$ -dimensional manifold  $\mathbf{V}$ , with  $n \geq 2$  and  $m \geq 1$  (for a number of physical applications, it will be considered to model a physical or geometric space). In a particular case,  $\mathbf{V} = TM$ , with  $n = m$  (i.e. a tangent bundle), or  $\mathbf{V} = \mathbf{E} = (E, M)$ ,  $\dim M = n$ , is a vector bundle on  $M$ , with total space  $E$  (we shall use such spaces for traditional definitions of Finsler and Lagrange spaces [37, 38, 33, 10, 53, 8, 54]). In a general case, a manifold  $\mathbf{V}$  is provided with a local fibred structure into conventional "horizontal" and "vertical" directions defined by a nonholonomic (nonintegrable) distribution with associated nonlinear connection (equivalently, nonholonomic frame) structure. Such nonholonomic manifolds will be used for modelling locally anisotropic structures in Einstein gravity and generalizations [61, 78, 79, 64, 77].

### 2.2.1 Nonlinear connections and N-adapted frames

We denote by  $\pi^\top : T\mathbf{V} \rightarrow TM$  the differential of a map  $\pi : \mathbf{V} \rightarrow M$  defined by fiber preserving morphisms of the tangent bundles  $T\mathbf{V}$  and  $TM$ . The kernel of  $\pi^\top$  is just the vertical subspace  $v\mathbf{V}$  with a related inclusion mapping  $i : v\mathbf{V} \rightarrow T\mathbf{V}$ . For simplicity, in this work we restrict our considerations for a fibred manifold  $\mathbf{V} \rightarrow M$  with constant rank  $\pi$ . In such cases, we can define connections and metrics on  $\mathbf{V}$  in usual form, but with the aim to "mimic" Finsler and Lagrange like structures (not on usual tangent bundles but on such nonholonomic manifolds) we shall also consider metrics, tensors and connections adapted to the fibred structure as it was elaborated in Finsler geometry (see below the concept of distinguished metric, section 2.2.2 and distinguished connection, section 2.2.3).

A **nonlinear connection (N-connection)**  $\mathbf{N}$  on a manifold  $\mathbf{V}$  is defined by the splitting on the left of an exact sequence

$$0 \rightarrow v\mathbf{V} \xrightarrow{i} T\mathbf{V} \rightarrow T\mathbf{V}/v\mathbf{V} \rightarrow 0,$$

i. e. by a morphism of submanifolds  $\mathbf{N} : T\mathbf{V} \rightarrow v\mathbf{V}$  such that  $\mathbf{N} \circ i$  is the unity in  $v\mathbf{V}$ .<sup>4</sup>

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<sup>4</sup>There is a proof (see, for instance, Ref. [38], Theorem 1.2, page 21) that for a vector

Locally, a N-connection is defined by its coefficients  $N_i^a(u)$ ,

$$\mathbf{N} = N_i^a(u) dx^i \otimes \frac{\partial}{\partial y^a}. \quad (8)$$

In an equivalent form, we can say that any N-connection is defined by a **Whitney sum** of conventional horizontal (h) space,  $(h\mathbf{V})$ , and vertical (v) space,  $(v\mathbf{V})$ ,

$$T\mathbf{V} = h\mathbf{V} \oplus v\mathbf{V}. \quad (9)$$

The sum (9) states on  $T\mathbf{V}$  a nonholonomic (equivalently, anholonomic, or nonintegrable) distribution of h- and v-space. The well known class of linear connections consists on a particular subclass with the coefficients being linear on  $y^a$ , i.e.

$$N_i^a(u) = \Gamma_{bj}^a(x) y^b. \quad (10)$$

The geometric objects on  $\mathbf{V}$  can be defined in a form adapted to a N-connection structure, following decompositions which are invariant under parallel transports preserving the splitting (9). In this case, we call them to be distinguished (by the N-connection structure), i.e. **d-objects**. For instance, a vector field  $\mathbf{X} \in T\mathbf{V}$  is expressed

$$\mathbf{X} = (hX, vX), \text{ or } \mathbf{X} = X^\alpha \mathbf{e}_\alpha = X^i \mathbf{e}_i + X^a e_a,$$

where  $hX = X^i \mathbf{e}_i$  and  $vX = X^a e_a$  state, respectively, the adapted to the N-connection structure horizontal (h) and vertical (v) components of the vector. In brief,  $\mathbf{X}$  is called a distinguished vectors, **d-vector**.<sup>5</sup> In a similar fashion, the geometric objects on  $\mathbf{V}$  like tensors, spinors, connections, ... are called respectively **d-tensors**, **d-spinors**, **d-connections** if they are adapted to the N-connection splitting (9).

The **N-connection curvature** is defined as the **Neijenhuis tensor**

$$\Omega(\mathbf{X}, \mathbf{Y}) \doteq [vX, vY] + v[\mathbf{X}, \mathbf{Y}] - v[vX, \mathbf{Y}] - v[\mathbf{X}, vY]. \quad (11)$$

In local form, we have for (11)  $\Omega = \frac{1}{2} \Omega_{ij}^a d^i \wedge d^j \otimes \partial_a$ , with coefficients

$$\Omega_{ij}^a = \frac{\partial N_i^a}{\partial x^j} - \frac{\partial N_j^a}{\partial x^i} + N_i^b \frac{\partial N_j^a}{\partial y^b} - N_j^b \frac{\partial N_i^a}{\partial y^b}. \quad (12)$$

Any N-connection  $\mathbf{N}$  may be characterized by an associated frame (vielbein) structure  $\mathbf{e}_\nu = (\mathbf{e}_i, e_a)$ , where

$$\mathbf{e}_i = \frac{\partial}{\partial x^i} - N_i^a(u) \frac{\partial}{\partial y^a} \text{ and } e_a = \frac{\partial}{\partial y^a}, \quad (13)$$

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bundle over a paracompact manifold there exist N-connections. In this work, we restrict our considerations only for fibered manifolds admitting N-connection and related N-adapted frame structures (see the end of this section).

<sup>5</sup>We shall use always "boldface" symbols if it would be necessary to emphasize that certain spaces and/or geometrical objects are provided/adapted to a N-connection structure, or with the coefficients computed with respect to N-adapted frames.

and the dual frame (coframe) structure  $\mathbf{e}^\mu = (e^i, e^a)$ , where

$$e^i = dx^i \text{ and } e^a = dy^a + N_i^a(u)dx^i, \quad (14)$$

see formulas (1). These vielbeins are called respectively **N-adapted frames and coframes**. In order to preserve a relation with the previous denotations [64, 77], we emphasize that  $\mathbf{e}_\nu = (\mathbf{e}_i, e_a)$  and  $\mathbf{e}^\mu = (e^i, e^a)$  are correspondingly the former "N-elongated" partial derivatives  $\delta_\nu = \delta/\partial u^\nu = (\delta_i, \partial_a)$  and N-elongated differentials  $\delta^\mu = \delta u^\mu = (d^i, \delta^a)$ . This emphasizes that the operators (13) and (14) define certain "N-elongated" partial derivatives and differentials which are more convenient for tensor and integral calculations on such nonholonomic manifolds. The vielbeins (14) satisfy the nonholonomy relations

$$[\mathbf{e}_\alpha, \mathbf{e}_\beta] = \mathbf{e}_\alpha \mathbf{e}_\beta - \mathbf{e}_\beta \mathbf{e}_\alpha = W_{\alpha\beta}^\gamma \mathbf{e}_\gamma \quad (15)$$

with (antisymmetric) nontrivial anholonomy coefficients  $W_{ia}^b = \partial_a N_i^b$  and  $W_{ji}^a = \Omega_{ij}^a$  defining a proper parametrization (for a  $n + m$  splitting by a N-connection  $N_i^a$ ) of (14).

### 2.2.2 N-anholonomic manifolds and d-metrics

For simplicity, we shall work with a particular class of nonholonomic manifolds: A manifold  $\mathbf{V}$  is **N-anholonomic** if its tangent space  $T\mathbf{V}$  is enabled with a N-connection structure (9).<sup>6</sup>

A **distinguished metric** (in brief, **d-metric**) on a N-anholonomic manifold  $\mathbf{V}$  is a usual second rank metric tensor  $\mathbf{g}$  which with respect to a N-adapted basis (14) can be written in the form

$$\mathbf{g} = g_{ij}(x, y) e^i \otimes e^j + h_{ab}(x, y) e^a \otimes e^b \quad (16)$$

defining a N-adapted decomposition  $\mathbf{g} = hg \oplus_N vg = [hg, vg]$ .

A **metric structure**  $\check{g}$  on a N-anholonomic manifold  $\mathbf{V}$  is a symmetric covariant second rank tensor field which is not degenerated and of constant signature in any point  $\mathbf{u} \in \mathbf{V}$ . Any metric on  $\mathbf{V}$ , with respect to a local coordinate basis  $du^\alpha = (dx^i, dy^a)$ , can be parametrized in the form

$$\check{g} = \underline{g}_{\alpha\beta}(u) du^\alpha \otimes du^\beta \quad (17)$$

where

$$\underline{g}_{\alpha\beta} = \begin{bmatrix} g_{ij} + N_i^a N_j^b h_{ab} & N_j^e h_{ae} \\ N_i^e h_{be} & h_{ab} \end{bmatrix}. \quad (18)$$

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<sup>6</sup>In a similar manner, we can consider different types of (super) spaces and low energy string limits [60, 62, 73, 59], Riemann or Riemann–Cartan manifolds [77], noncommutative bundles, or superbundles and gauge models [63, 74, 17, 75, 64], Clifford–Dirac spinor bundles and algebroids [65, 58, 61, 76, 79, 78], Lagrange–Fedosov manifolds [19]... provided with nonholonomic (super) distributions (9) and preferred systems of reference (supervielbeins).

Such a metric (18) is generic off-diagonal, i.e. it can not be diagonalized by coordinate transforms if  $N_i^a(u)$  are any general functions.

In general, a metric structure is not adapted to a N-connection structure, but we can transform it into a d-metric

$$\mathbf{g} = hg(hX, hY) + vg(vX, vY) \quad (19)$$

adapted to a N-connection structure defined by coefficients  $N_i^a$ . We introduce denotations  $h\check{g}(hX, hY) = hg(hX, hY)$  and  $v\check{g}(vX, vY) = vg(vX, vY)$  and try to find a N-connection when

$$\check{g}(hX, vY) = 0 \quad (20)$$

for any d-vectors  $\mathbf{X}, \mathbf{Y}$ . In local form, for  $hX \rightarrow e_i$  and  $vY \rightarrow e_a$ , the equation (20) is an algebraic equation for the N-connection coefficients  $N_i^a$ ,

$$\check{g}(e_i, e_a) = 0, \text{ equivalently, } \underline{g}_{ia} - N_i^b h_{ab} = 0, \quad (21)$$

where  $\underline{g}_{ia} \doteq g(\partial/\partial x^i, \partial/\partial y^a)$ , which allows us to define in a unique form the coefficients  $N_i^b = h^{ab} \underline{g}_{ia}$  where  $h^{ab}$  is inverse to  $h_{ab}$ . We can write the metric  $\check{g}$  with ansatz (18) in equivalent form, as a d-metric (16) adapted to a N-connection structure, if we define  $g_{ij} \doteq \mathbf{g}(e_i, e_j)$  and  $h_{ab} \doteq \mathbf{g}(e_a, e_b)$  and consider the vielbeins  $\mathbf{e}_\alpha$  and  $\mathbf{e}^\alpha$  to be respectively of type (13) and (14).

A metric  $\check{g}$  (17) can be equivalently transformed into a d-metric (16) by performing a frame (vielbein) transform

$$\mathbf{e}_\alpha = \mathbf{e}_\alpha^{\underline{a}} \partial_{\underline{a}} \text{ and } \mathbf{e}^\beta = \mathbf{e}^\beta_{\underline{b}} du^{\underline{b}}, \quad (22)$$

with coefficients

$$\mathbf{e}_\alpha^{\underline{a}}(u) = \begin{bmatrix} e_i^{\underline{a}}(u) & N_i^b(u) e_b^{\underline{a}}(u) \\ 0 & e_a^{\underline{a}}(u) \end{bmatrix}, \quad (23)$$

$$\mathbf{e}^\beta_{\underline{b}}(u) = \begin{bmatrix} e^i_{\underline{b}}(u) & -N_k^b(u) e^k_{\underline{b}}(u) \\ 0 & e^a_{\underline{b}}(u) \end{bmatrix}, \quad (24)$$

being linear on  $N_i^a$ .

It should be noted here that parametrizations of metrics of type (18) have been introduced in Kaluza-Klein gravity [46] for the case of linear connections (10) and compactified extra dimensions  $y^a$ . For the five (or higher) dimensions, the coefficients  $\Gamma_{bi}^a(x)$  were considered as Abelian or non-Abelian gauge fields. In our approach, the coefficients  $N_i^b(x, y)$  are general ones, not obligatory linearized and/or compactified on  $y^a$ . For some models of Finsler gravity, the values  $N_i^a$  were treated as certain generalized nonlinear gauge fields (see Appendix to Ref. [37]), or as certain objects

defining (semi) spray configurations in generalized Finsler and Lagrange gravity [37, 38, 3].

On N-anholonomic manifolds, we can say that the coordinates  $x^i$  are holonomic and the coordinates  $y^a$  are nonholonomic (on N-anholonomic vector bundles, such coordinates are called respectively to be the horizontal and vertical ones). We conclude that a N-anholonomic manifold  $\mathbf{V}$  provided with a metric structure  $\check{g}$  (17) (equivalently, with a d-metric (16)) is a usual manifold (in particular, a pseudo-Riemannian one) with a prescribed nonholonomic  $n + m$  splitting into conventional “horizontal” and “vertical” subspaces (9) induced by the “off-diagonal” terms  $N_i^b(u)$  and the corresponding preferred nonholonomic frame structure (15).

### 2.2.3 d-torsions and d-curvatures

From the general class of linear connections which can be defined on a manifold  $V$ , and any its N-anholonomic versions  $\mathbf{V}$ , we distinguish those which are adapted to a N-connection structure  $\mathbf{N}$ .

A **distinguished connection (d-connection)  $\mathbf{D}$**  on a N-anholonomic manifold  $\mathbf{V}$  is a linear connection conserving under parallelism the Whitney sum (9). For any d-vector  $\mathbf{X}$ , there is a decomposition of  $\mathbf{D}$  into h- and v-covariant derivatives,

$$\mathbf{D}\mathbf{X} \doteq \mathbf{X} \rfloor \mathbf{D} = h\mathbf{X} \rfloor \mathbf{D} + v\mathbf{X} \rfloor \mathbf{D} = D h\mathbf{X} + D v\mathbf{X} = hD\mathbf{X} + vD\mathbf{X}. \quad (25)$$

The symbol “ $\rfloor$ ” in (25) denotes the interior product defined by a metric (17) (equivalently, by a d-metric (16)). The N-adapted components  $\Gamma_{\beta\gamma}^\alpha$  of a d-connection  $\mathbf{D}_\alpha = (\mathbf{e}_\alpha \rfloor \mathbf{D})$  are defined by the equations

$$\mathbf{D}_\alpha \mathbf{e}_\beta = \Gamma_{\alpha\beta}^\gamma \mathbf{e}_\gamma, \text{ or } \Gamma_{\alpha\beta}^\gamma(u) = (\mathbf{D}_\alpha \mathbf{e}_\beta) \rfloor \mathbf{e}^\gamma. \quad (26)$$

The N-adapted splitting into h- and v-covariant derivatives is stated by

$$h\mathbf{D} = \{\mathbf{D}_k = (L_{jk}^i, L_{bk}^a)\}, \text{ and } v\mathbf{D} = \{\mathbf{D}_c = (C_{jc}^i, C_{bc}^a)\},$$

where  $L_{jk}^i = (\mathbf{D}_k \mathbf{e}_j) \rfloor \mathbf{e}^i$ ,  $L_{bk}^a = (\mathbf{D}_k \mathbf{e}_b) \rfloor \mathbf{e}^a$ ,  $C_{jc}^i = (\mathbf{D}_c \mathbf{e}_j) \rfloor \mathbf{e}^i$ ,  $C_{bc}^a = (\mathbf{D}_c \mathbf{e}_b) \rfloor \mathbf{e}^a$ . The components  $\Gamma_{\alpha\beta}^\gamma = (L_{jk}^i, L_{bk}^a, C_{jc}^i, C_{bc}^a)$  completely define a d-connection  $\mathbf{D}$  on a N-anholonomic manifold  $\mathbf{V}$ . We shall write conventionally that  $\mathbf{D} = (hD, vD)$ , or  $\mathbf{D}_\alpha = (D_i, D_a)$ , with  $hD = (L_{jk}^i, L_{bk}^a)$  and  $vD = (C_{jc}^i, C_{bc}^a)$ , see (26).

The **torsion and curvature** of a d-connection  $\mathbf{D} = (hD, vD)$ , **d-torsions and d-curvatures**, are defined similarly to formulas (4) and (5) with further h- and v-decompositions. The simplest way to perform computations with d-connections is to use **N-adapted differential forms** like

$$\Gamma_\beta^\alpha = \Gamma_{\beta\gamma}^\alpha \mathbf{e}^\gamma \quad (27)$$

with the coefficients defined with respect to (14) and (13). For instance, torsion can be computed in the form

$$\mathcal{T}^\alpha \doteq \mathbf{D}\mathbf{e}^\alpha = d\mathbf{e}^\alpha + \Gamma^\alpha_\beta \wedge \mathbf{e}^\beta. \quad (28)$$

Locally it is characterized by (N-adapted) d-torsion coefficients

$$\begin{aligned} T^i_{jk} &= L^i_{jk} - L^i_{kj}, \quad T^i_{ja} = -T^i_{aj} = C^i_{ja}, \quad T^a_{ji} = \Omega^a_{ji}, \\ T^a_{bi} &= -T^a_{ib} = \frac{\partial N^a_i}{\partial y^b} - L^a_{bi}, \quad T^a_{bc} = C^a_{bc} - C^a_{cb}. \end{aligned} \quad (29)$$

By a straightforward d-form calculus, we can compute the N-adapted components  $\mathbf{R} = \{\mathbf{R}^\alpha_{\beta\gamma\delta}\}$  of the curvature

$$\mathcal{R}^\alpha_\beta \doteq \mathbf{D}\Gamma^\alpha_\beta = d\Gamma^\alpha_\beta - \Gamma^\gamma_\beta \wedge \Gamma^\alpha_\gamma = \mathbf{R}^\alpha_{\beta\gamma\delta} \mathbf{e}^\gamma \wedge \mathbf{e}^\delta, \quad (30)$$

of a d-connection  $\mathbf{D}$ ,

$$\begin{aligned} R^i_{hjk} &= e_k L^i_{hj} - e_j L^i_{hk} + L^m_{hj} L^i_{mk} - L^m_{hk} L^i_{mj} - C^i_{ha} \Omega^a_{kj}, \\ R^a_{bjk} &= e_k L^a_{bj} - e_j L^a_{bk} + L^c_{bj} L^a_{ck} - L^c_{bk} L^a_{cj} - C^a_{bc} \Omega^c_{kj}, \\ R^i_{jka} &= e_a L^i_{jk} - D_k C^i_{ja} + C^i_{jb} T^b_{ka}, \\ R^c_{bka} &= e_a L^c_{bk} - D_k C^c_{ba} + C^c_{bd} T^d_{ka}, \\ R^i_{jbc} &= e_c C^i_{jb} - e_b C^i_{jc} + C^h_{jb} C^i_{hc} - C^h_{jc} C^i_{hb}, \\ R^a_{bcd} &= e_d C^a_{bc} - e_c C^a_{bd} + C^e_{bc} C^a_{ed} - C^e_{bd} C^a_{ec}. \end{aligned} \quad (31)$$

Contracting respectively the components of (31), one proves that the Ricci tensor  $\mathbf{R}_{\alpha\beta} \doteq \mathbf{R}^\tau_{\alpha\beta\tau}$  is characterized by h- v-components

$$R_{ij} \doteq R^k_{ijk}, \quad R_{ia} \doteq -R^k_{ika}, \quad R_{ai} \doteq R^b_{aib}, \quad R_{ab} \doteq R^c_{abc}. \quad (32)$$

It should be noted that this tensor is not symmetric for arbitrary d-connections  $\mathbf{D}$ , i.e.  $\mathbf{R}_{\alpha\beta} \neq \mathbf{R}_{\beta\alpha}$ . The **scalar curvature** of a d-connection is

$${}^s\mathbf{R} \doteq \mathbf{g}^{\alpha\beta} \mathbf{R}_{\alpha\beta} = g^{ij} R_{ij} + h^{ab} R_{ab}, \quad (33)$$

defined by a sum the h- and v-components of (32) and d-metric (16).

The Einstein d-tensor is defined and computed similarly to (7), but for d-connections,

$$\mathbf{E}_{\alpha\beta} = \mathbf{R}_{\alpha\beta} - \frac{1}{2} \mathbf{g}_{\alpha\beta} {}^s\mathbf{R} \quad (34)$$

This d-tensor defines an alternative to  ${}_h E_{\alpha\beta}$  (nonholonomic) Einstein configuration if its d-connection is defined in a unique form for an off-diagonal metric (18).

#### 2.2.4 Some classes of distinguished or non-adapted linear connections

From the class of arbitrary d-connections  $\mathbf{D}$  on  $\mathbf{V}$ , one distinguishes those which are **metric compatible (metrical d-connections)** satisfying the condition

$$\mathbf{D}\mathbf{g} = \mathbf{0} \quad (35)$$

including all h- and v-projections  $D_j g_{kl} = 0, D_a g_{kl} = 0, D_j h_{ab} = 0, D_a h_{bc} = 0$ . Different approaches to Finsler–Lagrange geometry modelled on  $\mathbf{TM}$  (or on the dual tangent bundle  $\mathbf{T}^*\mathbf{M}$ , in the case of Cartan–Hamilton geometry) were elaborated for different d-metric structures which are metric compatible [15, 37, 38, 35, 39, 36, 60, 62, 76] or not metric compatible [8].

For any d-metric  $\mathbf{g} = [hg, vg]$  on a N-anholonomic manifold  $\mathbf{V}$ , there is a unique metric canonical d-connection  $\hat{\mathbf{D}}$  satisfying the conditions  $\hat{\mathbf{D}}\mathbf{g} = 0$  and with vanishing  $h(hh)$ -torsion,  $v(vv)$ -torsion, i. e.  $h\hat{T}(hX, hY) = 0$  and  $v\hat{T}(vX, vY) = 0$ . By straightforward calculations, we can verify that  $\hat{\Gamma}^\gamma_{\alpha\beta} = (\hat{L}^i_{jk}, \hat{L}^a_{bk}, \hat{C}^i_{jc}, \hat{C}^a_{bc})$ , when

$$\begin{aligned} \hat{L}^i_{jk} &= \frac{1}{2}g^{ir}(e_k g_{jr} + e_j g_{kr} - e_r g_{jk}), \\ \hat{L}^a_{bk} &= e_b(N_k^a) + \frac{1}{2}h^{ac}(e_k h_{bc} - h_{dc} e_b N_k^d - h_{db} e_c N_k^d), \\ \hat{C}^i_{jc} &= \frac{1}{2}g^{ik}e_c g_{jk}, \quad \hat{C}^a_{bc} = \frac{1}{2}h^{ad}(e_c h_{bd} + e_b h_{cd} - e_d h_{bc}) \end{aligned} \quad (36)$$

result in  $\hat{T}^i_{jk} = 0$  and  $\hat{T}^a_{bc} = 0$  but  $\hat{T}^i_{ja}, \hat{T}^a_{ji}$  and  $\hat{T}^a_{bi}$  are not zero, see formulas (29) written for this canonical d-connection.

For any metric structure  $\mathbf{g}$  on a manifold  $\mathbf{V}$ , there is a unique metric compatible and torsionless **Levi Civita connection**  $\nabla = \{ \Gamma^\alpha_{\beta\gamma} \}$  for which  $\mathcal{T} = 0$  and  $\nabla g = 0$ . This is not a d-connection because it does not preserve under parallelism the N-connection splitting (9) (it is not adapted to the N-connection structure). Let us parametrize its coefficients in the form

$${}^i\Gamma^\alpha_{\beta\gamma} = ({}^iL^i_{jk}, {}^iL^a_{jk}, {}^iL^i_{bk}, {}^iL^a_{bk}, {}^iC^i_{jb}, {}^iC^a_{jb}, {}^iC^i_{bc}, {}^iC^a_{bc}),$$

where  $\nabla_{\mathbf{e}_k}(\mathbf{e}_j) = {}^iL^i_{jk}\mathbf{e}_i + {}^iL^a_{jk}e_a$ ,  $\nabla_{\mathbf{e}_k}(e_b) = {}^iL^i_{bk}\mathbf{e}_i + {}^iL^a_{bk}e_a$ ,  $\nabla_{e_b}(\mathbf{e}_j) = {}^iC^i_{jb}\mathbf{e}_i + {}^iC^a_{jb}e_a$ ,  $\nabla_{e_c}(e_b) = {}^iC^i_{bc}\mathbf{e}_i + {}^iC^a_{bc}e_a$ . A straightforward calculus<sup>7</sup>

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<sup>7</sup>Such results were originally considered by R. Miron and M. Anastasiei for vector bundles provided with N-connection and metric structures, see Ref. [38]. Similar proofs hold true for any nonholonomic manifold provided with a prescribed N-connection structure [77].



shows that the coefficients of the Levi-Civita connection are

$$\begin{aligned}
{}_1L_{jk}^i &= L_{jk}^i, \quad {}_1L_{jk}^a = -C_{jb}^i g_{ik} h^{ab} - \frac{1}{2} \Omega_{jk}^a, \\
{}_1L_{bk}^i &= \frac{1}{2} \Omega_{jk}^c h_{cb} g^{ji} - \frac{1}{2} (\delta_j^i \delta_k^h - g_{jk} g^{ih}) C_{hb}^j, \\
{}_1L_{bk}^a &= L_{bk}^a + \frac{1}{2} (\delta_c^a \delta_d^b + h_{cd} h^{ab}) [L_{bk}^c - e_b(N_k^c)], \\
{}_1C_{kb}^i &= C_{kb}^i + \frac{1}{2} \Omega_{jk}^a h_{cb} g^{ji} + \frac{1}{2} (\delta_j^i \delta_k^h - g_{jk} g^{ih}) C_{hb}^j, \\
{}_1C_{jb}^a &= -\frac{1}{2} (\delta_c^a \delta_b^d - h_{cb} h^{ad}) [L_{dj}^c - e_d(N_j^c)], \quad {}_1C_{bc}^a = C_{bc}^a, \\
{}_1C_{ab}^i &= -\frac{g^{ij}}{2} \{ [L_{aj}^c - e_a(N_j^c)] h_{cb} + [L_{bj}^c - e_b(N_j^c)] h_{ca} \},
\end{aligned} \tag{37}$$

where  $\Omega_{jk}^a$  are computed as in formula (12). For certain considerations, it is convenient to express

$${}_1\Gamma_{\alpha\beta}^\gamma = \widehat{\Gamma}_{\alpha\beta}^\gamma + {}_1Z_{\alpha\beta}^\gamma \tag{38}$$

where the explicit components of **distorsion tensor**  ${}_1Z_{\alpha\beta}^\gamma$  can be defined by comparing the formulas (37) and (36). It should be emphasized that all components of  ${}_1\Gamma_{\alpha\beta}^\gamma$ ,  $\widehat{\Gamma}_{\alpha\beta}^\gamma$  and  ${}_1Z_{\alpha\beta}^\gamma$  are uniquely defined by the coefficients of d-metric (16) and N-connection (8), or equivalently by the coefficients of the corresponding generic off-diagonal metric (18).

### 2.3 On equivalent (non)holonomic formulations of gravity theories

A **N-anholonomic Riemann-Cartan manifold**  ${}^{RC}\mathbf{V}$  is defined by a d-metric  $\mathbf{g}$  and a metric d-connection  $\mathbf{D}$  structures. We can say that a space  ${}^R\widehat{\mathbf{V}}$  is a canonical N-anholonomic Riemann manifold if its d-connection structure is canonical, i.e.  $\mathbf{D} = \widehat{\mathbf{D}}$ . The d-metric structure  $\mathbf{g}$  on  ${}^{RC}\mathbf{V}$  is of type (16) and satisfies the metricity conditions (35). With respect to a local coordinate basis, the metric  $\mathbf{g}$  is parametrized by a generic off-diagonal metric ansatz (18). For a particular case, we can treat the torsion  $\widehat{\mathbf{T}}$  as a nonholonomic frame effect induced by a nonintegrable N-splitting. We conclude that a manifold  ${}^R\widehat{\mathbf{V}}$  is enabled with a nontrivial torsion (29) (uniquely defined by the coefficients of N-connection (8), and d-metric (16) and canonical d-connection (36) structures). Nevertheless, such manifolds can be described alternatively, equivalently, as a usual (holonomic) Riemann manifold with the usual Levi Civita for the metric (17) with coefficients (18). We do not distinguish the existing nonholonomic structure for such geometric constructions.

Having prescribed a nonholonomic  $n + m$  splitting on a manifold  $V$ , we can define two canonical linear connections  $\nabla$  and  $\widehat{\mathbf{D}}$ . Correspondingly, these

connections are characterized by two curvature tensors,  ${}_1R^\alpha_{\beta\gamma\delta}(\nabla)$  (computed by introducing  ${}_1\Gamma^\alpha_{\beta\gamma}$  into (27) and (30)) and  $\mathbf{R}^\alpha_{\beta\gamma\delta}(\hat{\mathbf{D}})$  (with the N-adapted coefficients computed following formulas (31)). Contracting indices, we can commute the Ricci tensor  $Ric(\nabla)$  and the Ricci d-tensor  $\mathbf{Ric}(\hat{\mathbf{D}})$  following formulas (32), correspondingly written for  $\nabla$  and  $\hat{\mathbf{D}}$ . Finally, using the inverse d-tensor  $\mathbf{g}^{\alpha\beta}$  for both cases, we compute the corresponding scalar curvatures  ${}^sR(\nabla)$  and  ${}^s\mathbf{R}(\hat{\mathbf{D}})$ , see formulas (33) by contracting, respectively, with the Ricci tensor and Ricci d-tensor.

The standard formulation of the Einstein gravity is for the connection  $\nabla$ , when the field equations are written in the form (7). But it can be equivalently reformulated by using the canonical d-connection, or other connections uniquely defined by the metric structure. If a metric (18)  $\underline{g}_{\alpha\beta}$  is a solution of the Einstein equations  ${}_1E_{\alpha\beta} = \Upsilon_{\alpha\beta}$ , having prescribed a  $(n+m)$ -decomposition, we can define algebraically the coefficients of a N-connection,  $N^a_i$ , N-adapted frames  $e_\alpha$  (13) and  $e^\beta$  (14), and d-metric  $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$  (16). The next steps are to compute  $\hat{\Gamma}^\gamma_{\alpha\beta}$ , following formulas (36), and then using (31), (32) and (33) for  $\hat{\mathbf{D}}$ , to define  $\hat{\mathbf{E}}_{\alpha\beta}$  (34). The Einstein equations with matter sources, written in equivalent form by using the canonical d-connection, are

$$\hat{\mathbf{E}}_{\alpha\beta} = \Upsilon_{\alpha\beta} + {}^Z\Upsilon_{\alpha\beta}, \quad (39)$$

where the effective source  ${}^Z\Upsilon_{\alpha\beta}$  is just the deformation tensor of the Einstein tensor computed by introducing deformation (38) into the left part of (7); all decompositions being performed with respect to the N-adapted co-frame (14), when  ${}_1E_{\alpha\beta} = \hat{\mathbf{E}}_{\alpha\beta} - {}^Z\Upsilon_{\alpha\beta}$ . For certain matter field/ string gravity configurations, the solutions of (39) also solve the equations (7). Nevertheless, because of generic nonlinear character of gravity and gravity-matter field interactions and functions defining nonholonomic distributions, one could be certain special conditions when even vacuum configurations contain a different physical information if to compare with usual holonomic ones. We analyze some examples:

In our works [64, 77, 65], we investigated a series of exact solutions defining N-anholonomic Einstein spaces related to generic off-diagonal solutions in general relativity by such nonholonomic constraints when  $\mathbf{Ric}(\hat{\mathbf{D}}) = Ric(\nabla)$ , even  $\hat{\mathbf{D}} \neq \nabla$ .<sup>8</sup> In this case, for instance, the solutions of the Einstein equations with cosmological constant  $\lambda$ ,

$$\hat{\mathbf{R}}_{\alpha\beta} = \lambda \mathbf{g}_{\alpha\beta} \quad (40)$$

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<sup>8</sup>One should be emphasized here that different type of connections on N-anholonomic manifolds have different coordinate and frame transform properties. It is possible, for instance, to get equalities of coefficients for some systems of coordinates even the connections are very different. The transformation laws of tensors and d-tensors are also different if some objects are adapted and other are not adapted to a prescribed N-connection structure.

can be transformed into metrics for usual Einstein spaces with Levi Civita connection  $\nabla$ . The idea is that for certain general metric ansatz, see section 3.2, the equations (40) can be integrated in general form just for the connection  $\hat{\mathbf{D}}$  but not for  $\nabla$ . The nontrivial torsion components

$$\hat{T}_{ja}^i = -\hat{T}_{aj}^i = \hat{C}_{ja}^i, \quad \hat{T}_{ji}^a = \hat{T}_{ij}^a = \Omega_{ji}^a, \quad \hat{T}_{bi}^a = -\hat{T}_{ib}^a = \frac{\partial N_i^a}{\partial y^b} - \hat{L}_{bi}^a, \quad (41)$$

see (29), for some configurations, may be associated with an absolute anti-symmetric  $H$ -fields in string gravity [18, 51], but nonholonomically transformed to N-adapted bases, see details in [64, 77].

For more restricted configurations, we can search solutions with metric ansatz defining Einstein foliated spaces, when

$$\Omega_{jk}^c = 0, \quad \hat{L}_{bk}^c = e_b(N_k^c), \quad \hat{C}_{jb}^i = 0, \quad (42)$$

and the d-torsion components (41) vanish, but the N-adapted frame structure has, in general, nontrivial anholonomy coefficients, see (15). One present a special interest a less constrained configurations with  $\hat{T}_{jk}^c = \Omega_{jk}^c \neq 0$  when  $\mathbf{Ric}(\hat{\mathbf{D}}) = Ric(\nabla)$  and  $\hat{T}_{jk}^i = \hat{T}_{bc}^a = 0$ , for certain general ansatz  $\hat{T}_{ja}^i = 0$  and  $\hat{T}_{bi}^a = 0$ , but  $\hat{\mathbf{R}}_{\beta\gamma\delta}^\alpha \neq R_{\beta\gamma\delta}^\alpha$ . In such cases, we constrain the integral varieties of equations (40) in such a manner that we generate integrable or nonintegrable distributions on a usual Einstein space defined by  $\nabla$ . This is possible because if the conditions (42) are satisfied, the deformation tensor  ${}_1Z_{\alpha\beta}^\gamma = 0$ . For  $\lambda = 0$ , if  $n + m = 4$ , for corresponding signature, we get foliated vacuum configurations in general relativity.

For N-anholonomic manifolds  $\mathbf{V}^{n+n}$  of odd dimensions, when  $m = n$ , and if  $g_{ij} = h_{ij}$  (we identify correspondingly, the h- and v-indices), we can consider a canonical d-connection  $\hat{\mathbf{D}} = (h\hat{D}, v\hat{D})$  with the nontrivial coefficients with respect to  $\mathbf{e}_\nu$  and  $\mathbf{e}^\mu$  parametrized respectively  $\hat{\mathbf{T}}_{\beta\gamma}^\alpha = (\hat{L}_{jk}^i = \hat{L}_{bk}^a, \hat{C}_{jc}^i = \hat{C}_{bc}^a)$ ,<sup>9</sup> for

$$\hat{L}_{jk}^i = \frac{1}{2}g^{ih}(\mathbf{e}_k g_{jh} + \mathbf{e}_j g_{kh} - \mathbf{e}_h g_{jk}), \quad \hat{C}_{bc}^a = \frac{1}{2}g^{ae}(e_b g_{ec} + e_c g_{eb} - e_e g_{bc}), \quad (43)$$

defining the generalized Christoffel symbols. Such nonholonomic configurations can be used for modelling generalized Finsler-Lagrange, and particular cases, defined in Refs. [37, 38] for  $\mathbf{V}^{n+n} = \mathbf{TM}$ , see below section 3.1. There are only three classes of d-curvatures for the d-connection (43),

$$\begin{aligned} \hat{R}_{hjk}^i &= \mathbf{e}_k \hat{L}_{hj}^i - \mathbf{e}_j \hat{L}_{hk}^i + \hat{L}_{hj}^m \hat{L}_{mk}^i - \hat{L}_{hk}^m \hat{L}_{mj}^i - \hat{C}_{ha}^i \Omega_{kj}^a, \\ \hat{P}_{jka}^i &= e_a \hat{L}_{jk}^i - \hat{\mathbf{D}}_k \hat{C}_{ja}^i, \quad \hat{S}_{bcd}^a = e_d \hat{C}_{bc}^a - e_c \hat{C}_{bd}^a + \hat{C}_{bc}^e \hat{C}_{ed}^a - \hat{C}_{bd}^e \hat{C}_{ec}^a, \end{aligned} \quad (44)$$

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<sup>9</sup>the equalities of indices " $i = a$ " are considered in the form " $i = 1 = a = n + 1$ ,  $i = 2 = a = n + 2$ , ...  $i = n = a = n + n$ "

where all indices  $a, b, \dots, i, j, \dots$  run the same values and, for instance,  $C_{bc}^e \rightarrow C_{jk}^i, \dots$ . Such locally anisotropic configurations are not integrable if  $\Omega_{kj}^a \neq 0$ , even the d-torsion components  $\hat{T}_{jk}^i = 0$  and  $\hat{T}_{bc}^a = 0$ . We note that for geometric models on  $\mathbf{V}^{n+n}$ , or on  $\mathbf{TM}$ , with  $g_{ij} = h_{ij}$ , one writes, in brief,  $\hat{\Gamma}_{\beta\gamma}^\alpha = (\hat{L}_{jk}^i, \hat{C}_{bc}^a)$ , or, for more general d-connections,  $\Gamma_{\beta\gamma}^\alpha = (L_{jk}^i, C_{bc}^a)$ , see below section 3.1, on Lagrange and Finsler spaces.

### 3 Nonholonomic Deformations of Manifolds and Vector Bundles

This section will deal mostly with nonholonomic distributions on manifolds and vector/ tangent bundles and their nonholonomic deformations modelling, on Riemann and Riemann–Cartan manifolds, different types of generalized Finsler–Lagrange geometries.

#### 3.1 Finsler–Lagrange spaces and generalizations

The notion of Lagrange space was introduced by J. Kern [27] and elaborated in details by R. Miron’s school, see Refs. [37, 38, 34, 35, 39, 36], as a natural extension of Finsler geometry [15, 53, 33, 10] (see also Refs. [60, 62, 73, 64, 77], on Lagrange–Finsler super/noncommutative geometry). Originally, such geometries were constructed on tangent bundles, but they also can be modelled on N-anholonomic manifolds, for instance, as models for certain gravitational interactions with prescribed nonholonomic constraints deformed symmetries.

##### 3.1.1 Lagrange spaces

A **differentiable Lagrangian**  $L(x, y)$ , i.e. a fundamental Lagrange function, is defined by a map  $L : (x, y) \in TM \rightarrow L(x, y) \in \mathbb{R}$  of class  $\mathcal{C}^\infty$  on  $\widetilde{TM} = TM \setminus \{0\}$  and continuous on the null section  $0 : M \rightarrow TM$  of  $\pi$ . A regular Lagrangian has non-degenerate **Hessian**

$${}^L g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 L(x, y)}{\partial y^i \partial y^j}, \quad (45)$$

when  $\text{rank} |g_{ij}| = n$  and  ${}^L g^{ij}$  is the inverse matrix. A **Lagrange space** is a pair  $L^n = [M, L(x, y)]$  with  ${}^L g_{ij}$  being of fixed signature over  $\mathbf{V} = \widetilde{TM}$ .

One holds the result: The **Euler–Lagrange equations**  $\frac{d}{d\tau} \left( \frac{\partial L}{\partial y^i} \right) - \frac{\partial L}{\partial x^i} = 0$ , where  $y^i = \frac{dx^i}{d\tau}$  for  $x^i(\tau)$  depending on parameter  $\tau$ , are equivalent to the **“nonlinear” geodesic equations**  $\frac{d^2 x^a}{d\tau^2} + 2G^a(x^k, \frac{dx^b}{d\tau}) = 0$  defining paths

of a canonical **semispray**  $S = y^i \frac{\partial}{\partial x^i} - 2G^a(x, y) \frac{\partial}{\partial y^a}$ , where  $2G^i(x, y) = \frac{1}{2} {}^L g^{ij} \left( \frac{\partial^2 L}{\partial y^i \partial x^k} y^k - \frac{\partial L}{\partial x^i} \right)$ .

There exists on  $\mathbf{V} \simeq \widetilde{TM}$  a canonical N-connection

$${}^L N_j^a = \frac{\partial G^a(x, y)}{\partial y^j} \quad (46)$$

defined by the fundamental Lagrange function  $L(x, y)$ , which prescribes non-holonomic frame structures of type (13) and (14),  ${}^L \mathbf{e}_\nu = ({}^L \mathbf{e}_i, e_a)$  and  ${}^L \mathbf{e}^\mu = (e^i, {}^L \mathbf{e}^a)$ . One defines the canonical metric structure

$${}^L \mathbf{g} = {}^L g_{ij}(x, y) e^i \otimes e^j + {}^L g_{ij}(x, y) {}^L \mathbf{e}^i \otimes {}^L \mathbf{e}^j \quad (47)$$

constructed as a Sasaki type lift from  $M$  for  ${}^L g_{ij}(x, y)$ , see details in [86, 37, 38].

There is a unique canonical d-connection  ${}^L \widehat{\mathbf{D}} = (h {}^L \widehat{D}, v {}^L \widehat{D})$  with the coefficients  ${}^L \widehat{\Gamma}_{\beta\gamma}^\alpha = ({}^L \widehat{L}_{jk}^i, {}^L \widehat{C}_{bc}^a)$  computed by formulas (43) for the d-metric (47) with respect to  ${}^L \mathbf{e}_\nu$  and  ${}^L \mathbf{e}^\mu$ . All such geometric objects, including the corresponding to  ${}^L \widehat{\Gamma}_{\beta\gamma}^\alpha$ ,  ${}^L \mathbf{g}$  and  ${}^L N_j^a$  d-curvatures  ${}^L \widehat{\mathbf{R}}_{\beta\gamma\delta}^\alpha = ({}^L \widehat{R}_{hjk}^i, {}^L \widehat{P}_{jka}^i, {}^L \widehat{S}_{bcd}^a)$ , see (44), are completely defined by a Lagrange fundamental function  $L(x, y)$  for a nondegenerate  ${}^L g_{ij}$ .

We conclude that any regular Lagrange mechanics can be geometrized as a nonholonomic Riemann manifold  ${}^L \mathbf{V}$  equipped with the canonical N-connection  ${}^L N_j^a$  (46). This geometrization was performed in such a way that the N-connection is induced canonically by the semispray configurations subjected the condition that the generalized nonlinear geodesic equations are equivalent to the Euler-Lagrange equations for  $L$ . Such mechanical models and semispray configurations can be used for a study of certain classes of nonholonomic effective analogous of gravitational interactions. The approach can be extended for more general classes of effective metrics, then those parametrized by (47), see next sections. After Kern and Miron and Anastasiei works, it was elaborated the so-called "analogous gravity" approach [9] with similar ideas modelling related to continuous mechanics, condensed media.... It should be noted here, that the constructions for higher order generalized Lagrange and Hamilton spaces [34, 35, 39, 36] provided a comprehensive geometric formalism for analogous models in gravity, geometric mechanics, continuous media, nonhomogeneous optics etc etc.

### 3.1.2 Finsler spaces

Following the ideas of the Romanian school on Finsler-Lagrange geometry and generalizations, any Finsler space defined by a **fundamental Finsler function**  $F(x, y)$ , being homogeneous of type  $F(x, \lambda y) = |\lambda| F(x, y)$ , for nonzero  $\lambda \in \mathbb{R}$ , may be considered as a particular case of Lagrange space

when  $L = F^2$  (on different rigorous mathematical definitions of Finsler spaces, see [53, 33, 37, 38, 10, 8]; in our approach with applications to physics, we shall not constrain ourself with special signatures, smooth class conditions and special types of connections). Historically, the bulk of mathematicians worked in an inverse direction, by generalizing the constructions from the Cartan's approach to Finsler geometry in order to include into consideration regular Lagrange mechanical systems, or to define Finsler geometries with another type of nonlinear and linear connection structures. The Finsler geometry, in terms of the normal canonical d-connection (43), derived for respective  ${}^F g_{ij}$  and  ${}^F N_j^a$ , can be modelled as for the case of Lagrange spaces considered in the previous section: we have to change formally all labels  $L \rightarrow F$  and take into consideration possible conditions of homogeneity (or  $TM$ , see the monographs [37, 38]).

For generalized Finsler spaces, a N-connection can be stated by a general set of coefficients  $N_j^a$  subjected to certain nonholonomy conditions. Of course, working with homogeneous functions on a manifold  $V^{n+n}$ , we can model a Finsler geometry both on holonomic and nonholonomic Riemannian manifolds, or on certain types of Riemann–Cartan manifolds enabled with preferred frame structures  ${}^F \mathbf{e}_\nu = ({}^F \mathbf{e}_i, e_a)$  and  ${}^F \mathbf{e}^\mu = (e^i, {}^F \mathbf{e}^a)$ . Bellow, in the section 3.2, we shall discuss how certain type Finsler configurations can be derived as exact solutions in Einstein gravity. Such constructions allow us to argue that Finsler geometry is also very important in standard physics and that it was a big confusion to treat it only as a "sophisticated" generalization of Riemann geometry, on tangent bundles, with not much perspectives for modern physics.

In a number of works (see monographs [37, 38, 8]), it is emphasized that the first example of Finsler metric was considered in the famous inauguration thesis of B. Riemann [52], long time before P. Finsler [20]. Perhaps, this is a reason, for many authors, to use the term Riemann–Finsler geometry. Nevertheless, we would like to emphasize that a Finsler space is not completely defined only by a metric structure of type

$${}^F g_{ij} = \frac{1}{2} \frac{\partial^2 F}{\partial y^i \partial y^j} \quad (48)$$

originally considered on the vertical fibers of a tangent bundle. There are necessary additional conventions about metrics on a total Finsler space, N-connections and linear connections. This is the source for different approaches, definitions, constructions and ambiguities related to Finsler spaces and applications. Roughly speaking, different famous mathematicians, and their schools, elaborated their versions of Finsler geometries following some special purposes in geometry, mechanics and physics.

The first complete model of Finsler geometry exists due to E. Cartan [15] who in the 20-30th years of previous century elaborated the concepts of

vector bundles, Riemann–Cartan spaces with torsion, moving frames, developed the theory of spinors, Pfaff forms ... and (in coordinate form) operated with nonlinear connection coefficients. The Cartan’s constructions were performed with metric compatible linear connections which is very important for applications to standard models in physics.

Latter, there were proposed different models of Finsler spaces with metric not compatible linear connections. The most notable connections were those by L. Berwald, S. -S. Chern (re–discovered by H. Rund), H. Shimada and others (see details, discussions and bibliography in monographs [37, 38, 8, 53]). For d–connections of type (43), there are distinguished three cases of metric compatibility (compare with h- and v-projections of formula (35)): A Finsler connection  ${}^F\mathbf{D}_\alpha = ({}^F D_k, {}^F D_a)$  is called h–metric if  ${}^F D_i^F g_{ij} = 0$ ; it is called v–metric if  ${}^F D_a {}^F g_{ij} = 0$  and it is metrical if both conditions are satisfied.

Here, we note three of the most important Finsler d–connections having their special geometric and (possible) physical merits:

1. The **canonical Finsler connection**  ${}^F\hat{\mathbf{D}}$  is defined by formulas (43), but for  ${}^F g_{ij}$ , i.e. as  ${}^F\hat{\Gamma}_{\beta\gamma}^\alpha = \left( {}^F\hat{L}_{jk}^i, {}^F\hat{C}_{bc}^a \right)$ . This d–connection is metrical. For a special class of N–connections  ${}^C N_j^a(x^k, y^b) = y^k {}^C L_{jk}^i$ , we get the famous **Cartan connection** for Finsler spaces,  ${}^C\Gamma_{\beta\gamma}^\alpha = \left( {}^C L_{jk}^i, {}^C C_{bc}^a \right)$ , with

$$\begin{aligned} {}^C L_{jk}^i &= \frac{1}{2} {}^F g^{ih} ({}^C \mathbf{e}_k {}^F g_{jh} + {}^C \mathbf{e}_j {}^F g_{kh} - {}^C \mathbf{e}_h {}^F g_{jk}), \\ {}^C C_{bc}^a &= \frac{1}{2} {}^F g^{ae} (e_b {}^F g_{ec} + e_c {}^F g_{eb} - e_e {}^F g_{bc}), \end{aligned} \quad (49)$$

where  ${}^C \mathbf{e}_k = \frac{\partial}{\partial x^k} - {}^C N_j^a \frac{\partial}{\partial y^a}$  and  $e_b = \frac{\partial}{\partial y^b}$ , which can be defined in a unique axiomatic form [33]. Such canonical and Cartan–Finsler connections, being metric compatible, for nonholonomic geometric models with local anisotropy on Riemann or Riemann–Cartan manifolds, are more suitable with the paradigm of modern standard physics.

2. The **Berwald connection**  ${}^B\mathbf{D}$  was introduced in the form  ${}^B\Gamma_{\beta\gamma}^\alpha = \left( \frac{\partial {}^C N_j^b}{\partial y^a}, 0 \right)$  [13]. This d–connection is defined completely by the N–connection structure but it is not metric compatible, both not h–metric and not v–metric.
3. The **Chern connection**  ${}^{Ch}\mathbf{D}$  was considered as a minimal Finsler extension of the Levi Civita connection,  ${}^{Ch}\Gamma_{\beta\gamma}^\alpha = \left( {}^C L_{jk}^i, 0 \right)$ , with  ${}^C L_{jk}^i$  defined as in (49), preserving the torsionless condition, being h–metric but not v–metric. It is an interesting case of nonholonomic

geometries when torsion is completely transformed into nonmetricity which for physicists presented a substantial interest in connection to the Weyl nonmetricity introduced as a method of preserving conformal symmetry of certain scalar field constructions in general relativity, see discussion in [22]. Nevertheless, it should be noted that the constructions with the Chern connection, in general, are not metric compatible and can not be applied in direct form to standard models of physics.

It should be noted that all mentioned types of d-connections are uniquely defined by the coefficients of Finsler type d-metric and N-connection structure (equivalently, by the coefficients of corresponding generic off-diagonal metric of type (18)) following well defined geometric conditions. From such d-connections, we can always 'extract' the Levi Civita connection, using formulas of type (37) and (38), and work in 'non-adapted' (to N-connection) form. From geometric point of view, we can work with all types of Finsler connections and elaborate equivalent approaches even different connections have different merits in some directions of physics. For instance, in [37, 38], there are considered the Kawaguchi metrization procedure and the Miron's method of computing all metric compatible Finsler connections starting with a canonical one. It was analyzed also the problem of transforming one Finsler connection into different ones on tangent bundles and the formalism of mutual transforms of connections was reconsidered for nonholonomic manifolds, see details in [77].

Different models of Finsler spaces can be elaborated in explicit form for different types of d-metrics, N-connections and d-connections. For instance, for a Finsler Hessian (48) defining a particular case of d-metrics (47), or (16), denoted  ${}^F\mathbf{g}$ , for any type of connection (for instance, canonical d-connection, Cartan-Finsler, Berwald, Chern etc), we can compute the curvatures by using formulas (44) when "hat" labels are changed into the corresponding ones " $C, B, Ch, \dots$ ". This way, we model Finsler geometries on tangent bundles, like it is considered in the bulk of monographs [15, 33, 53, 37, 38, 10, 8], or on nonholonomic manifolds [80, 81, 82, 23, 11, 77].

With the aim to develop new applications in standard models of physics, let say in classical general relativity, when Finsler like structures are modelled on a (pseudo) Riemannian manifold (we shall consider explicit examples in the next sections), it is positively sure that the canonical Finsler and Cartan connections, and their variants of canonical d-connection on vector bundles and nonholonomic manifolds, should be preferred for constructing new classes of Einstein spaces and defining certain low energy limits to locally anisotropic string gravity models. Here we note that it is a very difficult problem to define Finsler-Clifford spaces with Finsler spinors, noncommutative generalizations to supersymmetric/ noncommutative Finsler geometry if we work with nonmetric d-connections, see discussions in [77, 76, 62].

We cite a proof [8] that any Lagrange fundamental function  $L$  can be



modelled as a singular case for a certain class of Finsler geometries of extra dimension (perhaps, the authors were oriented to prove from a mathematical point of view that it is not necessary to develop Finsler geometry as a new theory for Lagrange spaces, or their dual constructions for the Hamilton spaces). This idea, together with the method of Kawaguchi–Miron transforms of connections, can be related to the H. Poincare philosophical concepts about conventionality of the geometric space and field interaction theories [49, 50]. According to the Poincare’s geometry–physics dualism, the procedure of choosing a geometric arena for a physical theory is a question of convenience for researches to classify scientific data in an economical way, but not an action to be verified in physical experiments: as a matter of principle, any physical theory can be equivalently described on various types of geometric spaces by using more or less ”simple” geometric objects and transforms.

Nevertheless, the modern physics paradigm is based on the ideas of objective reality of physical laws and their experimental and theoretical verifications, at least in indirect form. The concept of Lagrangian is a very important geometrical and physical one and we shall distinguish the cases when we model a Lagrange or a Finsler geometry. A physical or mechanical model with a Lagrangian is not only a ”singular” case for a Finsler geometry but reflects a proper set of concepts, fundamental physical laws and symmetries describing real physical effects. We use the terms Finsler and Lagrange spaces in order to emphasize that they are different both from geometric and physical points of view. Certain geometric concepts and methods (like the N-connection geometry and nonholonomic frame transforms ...) are very important for both types of geometries, modelled on tangent bundles or on nonholonomic manifolds. This will be noted when we use the term Finsler–Lagrange geometry (structures, configurations, spaces).

One should be emphasized that the author of this review should not be considered as a physicist who does not accept nonmetric geometric constructions in modern physics. For instance, the Part I in monograph [77] is devoted to a deep study of the problem when generalized Finsler–Lagrange structures can be modelled on metric–affine spaces, even as exact solutions in gravity with nonmetricity [22], and, inversely, the Lagrange–affine and Finsler–affine spaces are classified by nonholonomic structures on metric–affine spaces. It is a question of convention on the type of physical theories one models by geometric methods. The standard theories of physics are formulated for metric compatible geometries, but further developments in quantum gravity may request certain type of nonmetric Finsler like geometries, or more general constructions. This is a topic for further investigations.

### 3.1.3 Generalized Lagrange spaces

There are various application in optics of nonhomogeneous media and gravity (see, for instance, Refs. [38, 77, 19, 64]) considering metrics of type  $g_{ij} \sim e^{\lambda(x,y)} {}^L g_{ij}(x, y)$  which can not be derived directly from a mechanical Lagrangian. The ideas and methods to work with arbitrary symmetric and nondegenerated tensor fields  $g_{ij}(x, y)$  were concluded in geometric and physical models for generalized Lagrange spaces, denoted  $GL^n = (M, g_{ij}(x, y))$ , on  $\widetilde{TM}$ , see [37, 38], where  $g_{ij}(x, y)$  is called the **fundamental tensor field**. Of course, the geometric constructions will be equivalent if we shall work on N-anholonomic manifolds  $\mathbf{V}^{n+n}$  with nonholonomic coordinates  $y$ . If we prescribe an arbitrary N-connection  $N_i^a(x, y)$  and consider that a metric  $g_{ij}$  defines both the h- and -v-components of a d-metric (16), we can introduce the canonical d-connection (43) and compute the components of d-curvature (44), define Ricci and Einstein tensors, elaborate generalized Lagrange models of gravity.

If we work with a general fundamental tensor field  $g_{ij}$  which can not be transformed into  ${}^L g_{ij}$ , we can consider an effective Lagrange function <sup>10</sup>,  $\mathcal{L}(x, y) \doteq g_{ab}(x, y)y^a y^b$  and use

$$\mathcal{L}_{g_{ab}} \doteq \frac{1}{2} \frac{\partial^2 \mathcal{L}}{\partial y^a \partial y^b} \quad (50)$$

as a Lagrange Hessian (47). A space  $GL^n = (M, g_{ij}(x, y))$  is said to be with a weakly regular metric if  $L^n = \left[ M, L = \sqrt{|\mathcal{L}|} \right]$  is a Lagrange space. For such spaces, we can define a canonical nonlinear connection structure

$$\mathcal{L} N_j^a(x, y) \doteq \frac{\partial \mathcal{L} G^a}{\partial y^j}, \quad (51)$$

$$\mathcal{L} G^a = \frac{1}{4} \mathcal{L} g^{ab} \left( y^k \frac{\partial \mathcal{L}}{\partial y^b \partial x^k} - \frac{\partial \mathcal{L}}{\partial x^a} \right) = \frac{1}{4} \mathcal{L} g^{ab} \left( \frac{\partial g_{bc}}{\partial y^d} + \frac{\partial g_{bd}}{\partial y^c} - \frac{\partial g_{cd}}{\partial y^b} \right) y^c y^d,$$

which allows us to write  $\mathcal{L} N_j^a$  is terms of the fundamental tensor field  $g_{ij}(x, y)$ . The geometry of such generalized Lagrange spaces is completely similar to that of usual Lagrange one, with that difference that we start not with a Lagrangian but with a fundamental tensor field.

In our papers [67, 1], we considered nonholonomic transforms of a metric  $\mathcal{L} g_{a'b'}(x, y)$

$$g_{ab}(x, y) = e_a^{a'}(x, y) e_b^{b'}(x, y) \mathcal{L} g_{a'b'}(x, y) \quad (52)$$

when  $\mathcal{L} g_{a'b'} \doteq \frac{1}{2} (e_{a'} e_{b'} \mathcal{L} + e_{b'} e_{a'} \mathcal{L}) = {}^0 g_{a'b'}$ , for  $e_{a'} = e_{a'}^a(x, y) \frac{\partial}{\partial y^a}$ , where  ${}^0 g_{a'b'}$  are constant coefficients (or in a more general case, they should result in a constant matrix for the d-curvatures (31) of a canonical d-connection

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<sup>10</sup>in [37, 38], it is called the absolute energy of a  $GL^n$ -space, but for further applications in modern gravity the term "energy" may result in certain type ambiguities

(36)). Such constructions allowed to derive proper solitonic hierarchies and bi-Hamilton structures for any (pseudo) Riemannian or generalized Finsler-Lagrange metric. The point was to work not with the Levi Civita connection (for which the solitonic equations became very cumbersome) but with a correspondingly defined canonical d-connection allowing to apply well defined methods from the geometry of nonlinear connections. Having encoded the "gravity and geometric mechanics" information into solitonic hierarchies and convenient d-connections, the constructions were shown to hold true if they are "inverted" to those with usual Levi Civita connections.

### 3.2 An ansatz for constructing exact solutions

We consider a four dimensional (4D) manifold  $\mathbf{V}$  of necessary smooth class and conventional splitting of dimensions  $\dim \mathbf{V} = n+m$  for  $n = 2$  and  $m = 2$ . The local coordinates are labelled in the form  $u^\alpha = (x^i, y^a) = (x^1, x^2, y^3 = v, y^4)$ , for  $i = 1, 2$  and  $a, b, \dots = 3, 4$ .

The ansatz of type (16) is parametrized in the form

$$\begin{aligned} \mathbf{g} &= g_1(x^i)dx^1 \otimes dx^1 + g_2(x^i)dx^2 \otimes dx^2 \\ &\quad + h_3(x^k, v) \delta v \otimes \delta v + h_4(x^k, v) \delta y \otimes \delta y, \\ \delta v &= dv + w_i(x^k, v) dx^i, \quad \delta y = dy + n_i(x^k, v) dx^i \end{aligned} \quad (53)$$

with the coefficients defined by some necessary smooth class functions of type  $g_{1,2} = g_{1,2}(x^1, x^2)$ ,  $h_{3,4} = h_{3,4}(x^i, v)$ ,  $w_i = w_i(x^k, v)$ ,  $n_i = n_i(x^k, v)$ . The off-diagonal terms of this metric, written with respect to the coordinate dual frame  $du^\alpha = (dx^i, dy^a)$ , can be redefined to state a N-connection structure  $\mathbf{N} = [N_i^3 = w_i(x^k, v), N_i^4 = n_i(x^k, v)]$  with a N-elongated co-frame (14) parametrized as

$$e^1 = dx^1, \quad e^2 = dx^2, \quad \mathbf{e}^3 = \delta v = dv + w_i dx^i, \quad \mathbf{e}^4 = \delta y = dy + n_i dx^i. \quad (54)$$

This vielbein is dual to the local basis

$$\mathbf{e}_i = \frac{\partial}{\partial x^i} - w_i(x^k, v) \frac{\partial}{\partial v} - n_i(x^k, v) \frac{\partial}{\partial y^5}, \quad e_3 = \frac{\partial}{\partial v}, \quad e_4 = \frac{\partial}{\partial y^5}, \quad (55)$$

which is a particular case of the N-adapted frame (13). The metric (53) does not depend on variable  $y^4$ , i.e. it possesses a Killing vector  $e_4 = \partial/\partial y^4$ , and distinguish the dependence on the so-called "anisotropic" variable  $y^3 = v$ .

Computing the components of the Ricci and Einstein tensors for the metric (53) and canonical d-connection (see details on tensors components' calculus in Refs. [68, 77]), one proves that the Einstein equations (39) for a diagonal with respect to (54) and (55) source,

$$\Upsilon_\beta^\alpha + {}^Z \Upsilon_\beta^\alpha = [\Upsilon_1^1 = \Upsilon_2(x^i, v), \Upsilon_2^2 = \Upsilon_2(x^i, v), \Upsilon_3^3 = \Upsilon_4(x^i), \Upsilon_4^4 = \Upsilon_4(x^i)] \quad (56)$$

transform into this system of partial differential equations:

$$\widehat{R}_1^1 = \widehat{R}_2^2 \quad (57)$$

$$= \frac{1}{2g_1g_2} \left[ \frac{g_1^\bullet g_2^\bullet}{2g_1} + \frac{(g_2^\bullet)^2}{2g_2} - g_2^{\bullet\bullet} + \frac{g_1' g_2'}{2g_2} + \frac{(g_1')^2}{2g_1} - g_1'' \right] = -\Upsilon_4(x^i),$$

$$\widehat{S}_3^3 = \widehat{S}_4^4 = \frac{1}{2h_3h_4} \left[ h_4^* \left( \ln \sqrt{|h_3h_4|} \right)^* - h_4^{**} \right] = -\Upsilon_2(x^i, v), \quad (58)$$

$$\widehat{R}_{3i} = -w_i \frac{\beta}{2h_4} - \frac{\alpha_i}{2h_4} = 0, \quad (59)$$

$$\widehat{R}_{4i} = -\frac{h_3}{2h_4} [n_i^{**} + \gamma n_i^*] = 0, \quad (60)$$

where, for  $h_{3,4}^* \neq 0$ ,

$$\alpha_i = h_4^* \partial_i \phi, \quad \beta = h_4^* \phi^*, \quad \gamma = \frac{3h_4^*}{2h_4} - \frac{h_3^*}{h_3}, \quad (61)$$

$$\phi = \ln |h_3^* / \sqrt{|h_3h_4|}|, \quad (62)$$

when the necessary partial derivatives are written in the form  $a^\bullet = \partial a / \partial x^1$ ,  $a' = \partial a / \partial x^2$ ,  $a^* = \partial a / \partial v$ . In the vacuum case, we must consider  $\Upsilon_{2,4} = 0$ . We note that we use a source of type (56) in order to show that the anholonomic frame method can be applied also for non-vacuum solutions, for instance, when  $\Upsilon_2 = \lambda_2 = \text{const}$  and  $\Upsilon_4 = \lambda_4 = \text{const}$ , defining locally anisotropic configurations generated by an anisotropic cosmological constant, which in its turn, can be induced by certain ansatz for the so-called  $H$ -field (absolutely antisymmetric third rank tensor field) in string theory [64, 77, 68]. Here we note that the off-diagonal gravitational interactions can model locally anisotropic configurations even if  $\lambda_2 = \lambda_4$ , or both values vanish.

In string gravity, the nontrivial torsion components and source  $\kappa \Upsilon_{\alpha\beta}$  can be related to certain effective interactions with the strength (torsion)

$$H_{\mu\nu\rho} = \mathbf{e}_\mu B_{\nu\rho} + \mathbf{e}_\rho B_{\mu\nu} + \mathbf{e}_\nu B_{\rho\mu}$$

of an antisymmetric field  $B_{\nu\rho}$ , when

$$R_{\mu\nu} = -\frac{1}{4} H_\mu^{\nu\rho} H_{\nu\lambda\rho} \quad (63)$$

$$D_\lambda H^{\lambda\mu\nu} = 0, \quad (64)$$

see details on string gravity, for instance, in Refs. [18, 51]. The conditions (63) and (64) are satisfied by the ansatz

$$H_{\mu\nu\rho} = \widehat{Z}_{\mu\nu\rho} + \widehat{H}_{\mu\nu\rho} = \lambda_{[H]} \sqrt{|g_{\alpha\beta}|} \varepsilon_{\nu\lambda\rho} \quad (65)$$

where  $\varepsilon_{\nu\lambda\rho}$  is completely antisymmetric and the distortion (from the Levi-Civita connection) and  $\widehat{Z}_{\mu\alpha\beta}\mathbf{e}^\mu = \mathbf{e}_\beta\rfloor\mathcal{T}_\alpha - \mathbf{e}_\alpha\rfloor\mathcal{T}_\beta + \frac{1}{2}(\mathbf{e}_\alpha\rfloor\mathbf{e}_\beta\rfloor\mathcal{T}_\gamma)\mathbf{e}^\gamma$  is defined by the torsion tensor (28). Our  $H$ -field ansatz is different from those already used in string gravity when  $\widehat{H}_{\mu\nu\rho} = \lambda_{[H]}\sqrt{|g_{\alpha\beta}|}\varepsilon_{\nu\lambda\rho}$ . In our approach, we define  $H_{\mu\nu\rho}$  and  $\widehat{Z}_{\mu\nu\rho}$  from the respective ansatz for the  $H$ -field and nonholonomically deformed metric, compute the torsion tensor for the canonical distinguished connection and, finally, define the 'deformed'  $H$ -field as  $\widehat{H}_{\mu\nu\rho} = \lambda_{[H]}\sqrt{|g_{\alpha\beta}|}\varepsilon_{\nu\lambda\rho} - \widehat{Z}_{\mu\nu\rho}$ .

Summarizing the results for an ansatz (53) with arbitrary signatures  $\epsilon_\alpha = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$  (where  $\epsilon_\alpha = \pm 1$ ) and  $h_3^* \neq 0$  and  $h_4^* \neq 0$ , one proves [64, 68, 77] that any off-diagonal metric

$$\begin{aligned} \circ\mathbf{g} &= e^{\psi(x^i)} [\epsilon_1 dx^1 \otimes dx^1 + \epsilon_2 dx^2 \otimes dx^2] + \epsilon_3 h_0^2(x^i) \times \\ & [f^*(x^i, v)]^2 |\varsigma(x^i, v)| \delta v \otimes \delta v + \epsilon_4 [f(x^i, v) - f_0(x^i)]^2 \delta y^4 \otimes \delta y^4, \\ \delta v &= dv + w_k(x^i, v) dx^k, \quad \delta y^4 = dy^4 + n_k(x^i, v) dx^k, \end{aligned} \quad (66)$$

where  $\psi(x^i)$  is a solution of the 2D equation  $\epsilon_1 \psi^{\bullet\bullet} + \epsilon_2 \psi'' = \Upsilon_4$ ,

$$\varsigma(x^i, v) = \varsigma_{[0]}(x^i) - \frac{\epsilon_3}{8} h_0^2(x^i) \int \Upsilon_2(x^k, v) f^*(x^i, v) [f(x^i, v) - f_0(x^i)] dv,$$

for a given source  $\Upsilon_4(x^i)$ , and the N-connection coefficients  $N_i^3 = w_i(x^k, v)$  and  $N_i^4 = n_i(x^k, v)$  are computed following the formulas

$$w_i = -\frac{\partial_i \varsigma(x^k, v)}{\varsigma^*(x^k, v)} \quad (67)$$

$$n_k = {}^1n_k(x^i) + {}^2n_k(x^i) \int \frac{[f^*(x^i, v)]^2}{[f(x^i, v) - f_0(x^i)]^3} \varsigma(x^i, v) dv, \quad (68)$$

define an exact solution of the system of Einstein equations (57)–(60). It should be emphasized that such solutions depend on arbitrary nontrivial functions  $f(x^i, v)$  (with  $f^* \neq 0$ ),  $f_0(x^i)$ ,  $h_0^2(x^i)$ ,  $\varsigma_{[0]}(x^i)$ ,  ${}^1n_k(x^i)$  and  ${}^2n_k(x^i)$ , and sources  $\Upsilon_2(x^k, v)$ ,  $\Upsilon_4(x^i)$ . Such values for the corresponding signatures  $\epsilon_\alpha = \pm 1$  have to be defined by certain boundary conditions and physical considerations. These classes of solutions depending on integration functions are more general than those for diagonal ansatz depending, for instance, on one radial variable like in the case of the Schwarzschild solution (when the Einstein equations are reduced to an effective nonlinear ordinary differential equation, ODE). In the case of ODE, the integral varieties depend on integration constants which can be defined from certain boundary/asymptotic and symmetry conditions, for instance, from the constraint that far away from the horizon the Schwarzschild metric contains corrections from the Newton potential. Because the ansatz (53) results in a system of nonlinear partial differential equations (57)–(60), the solutions depend not only on integration constants, but on very general classes of integration functions.

The ansatz of type (53) with  $h_3^* = 0$  but  $h_4^* \neq 0$  (or, inversely,  $h_3^* \neq 0$  but  $h_4^* = 0$ ) consist more special cases and request a bit different method of constructing exact solutions. Nevertheless, such type solutions are also generic off-diagonal and they may be of substantial interest (the length of paper does not allow to include an analysis of such particular cases).

A very general class of exact solutions of the Einstein equations with nontrivial sources (56), in general relativity, is defined by the ansatz

$$\begin{aligned} \circ \mathbf{g} &= e^{\psi(x^i)} [\epsilon_1 dx^1 \otimes dx^1 + \epsilon_2 dx^2 \otimes dx^2] \\ &\quad + h_3(x^i, v) \delta v \otimes \delta v + h_4(x^i, v) \delta y^4 \otimes \delta y^4, \\ \delta v &= dv + w_1(x^i, v) dx^1 + w_2(x^i, v) dx^2, \\ \delta y^4 &= dy^4 + n_1(x^i) dx^1 + n_2(x^i) dx^2, \end{aligned} \quad (69)$$

with the coefficients restricted to satisfy the conditions

$$\begin{aligned} \epsilon_1 \psi^{\bullet\bullet} + \epsilon_2 \psi'' &= \Upsilon_4, \quad h_4^* \phi / h_3 h_4 = \Upsilon_2, \\ w'_1 - w_2^\bullet + w_2 w_1^* - w_1 w_2^* &= 0, \quad n'_1 - n_2^\bullet = 0, \end{aligned} \quad (70)$$

for  $w_i = \partial_i \phi / \phi^*$ , see (62), for given sources  $\Upsilon_4(x^k)$  and  $\Upsilon_2(x^k, v)$ . We note that the second equation in (70) relates two functions  $h_3$  and  $h_4$  and the third and forth equations satisfy the conditions (42).

Even the ansatz (53) depends on three coordinates  $(x^k, v)$ , it allows us to construct more general classes of solutions for d-metrics, depending on four coordinates: such solutions can be related by chains of nonholonomic transforms. New classes of generic off-diagonal solutions will describe nonholonomic Einstein spaces related to string gravity, if one of the chain metric is of type (66), or in Einstein gravity, if one of the chain metric is of type (69).

## 4 Einstein Gravity and Lagrange–Kähler Spaces

We show how nonholonomic Riemannian spaces can be transformed into almost Hermitian manifolds enabled with nonintegrable almost complex structures.

### 4.1 Almost Hermitian connections and general relativity

We prove that the Einstein gravity on a (pseudo) Riemannian manifold  $V^{n+n}$  can be equivalently redefined as an almost Hermitian model if a nonintegrable N-connection splitting is prescribed. The Einstein theory can be also modified by considering certain canonical lifts on tangent bundles. The first class of Finsler–Lagrange like models [71] preserves the local Lorentz symmetry and can be applied for constructing exact solutions in Einstein gravity or for developing some approaches to quantum gravity following methods of geometric/deformation quantization. The second class of such

models [70] can be considered for some extensions to canonical quantum theories of gravity which can be elaborated in a renormalizable form, but, in general, result in violation of local Lorentz symmetry by such quantum effects.

#### 4.1.1 Nonholonomic deformations in Einstein gravity

Let us consider a metric  $\underline{g}_{\alpha\beta}$  (18), which for a  $(n+n)$ -splitting by a set of prescribed coefficients  $N_i^a(x, y)$  can be represented as a d-metric  $\mathbf{g}$  (16). Respectively, we can write the Einstein equations in the form (7), or, equivalently, in the form (39) with the source  ${}^Z\Upsilon_{\alpha\beta}$  defined by the off-diagonal metric coefficients of  $\underline{g}_{\alpha\beta}$ , depending linearly on  $N_i^a$ , and generating the distortion tensor  ${}_iZ^\gamma_{\alpha\beta}$ .

Computing the Ricci and Einstein d-tensors, we conclude that the Einstein equations written in terms of the almost Hermitian d-connection can be also parametrized in the form (39). Such geometric structures are non-holonomic: working respectively with  $\underline{g}$ ,  $\mathbf{g}$ , we elaborate equivalent geometric and physical models on  $V^{n+n}$ ,  $\mathbf{V}^{n+n}$ . Even for vacuum configurations, when  $\Upsilon_{\alpha\beta} = 0$ , in the almost Hermitian model of the Einstein gravity, we have an effective source  ${}^Z\Upsilon_{\alpha\beta}$  induced by the coefficients of generic off-diagonal metric. Nevertheless, there are possible integrable configurations, when the conditions (42) are satisfied. In this case,  ${}^Z\Upsilon_{\alpha\beta} = 0$ , and we can construct effective Hermitian configurations defining vacuum Einstein foliations.

One should be noted that the geometry of nonholonomic 2+2 splitting in general relativity, with nonholonomic frames and d-connections, or almost Hermitian connections, is very different from the geometry of the well known 3+1 splitting ADM formalism, see [40], when only the Levi Civita connection is used. Following the anholonomic frame method, we work with different classes of connections and frames when some new symmetries and invariants are distinguished and the field equations became exactly integrable for some general metric ansatz. Constraining or redefining the integral varieties and geometric objects, we can generate, for instance, exact solutions in Einstein gravity and compute quantum corrections to such solutions.

#### 4.1.2 Conformal lifts of Einstein structures to tangent bundles

Let us consider a pseudo-Riemannian manifold  $M$  enabled with a metric  ${}_ig_{ij}(x)$  as a solution of the Einstein equations. We define a procedure lifting  ${}_ig_{ij}(x)$  conformally on  $TM$  and inducing a generalized Lagrange structure and a corresponding almost Hermitian geometry. Let us introduce  ${}^\varpi\mathcal{L}(x, y) \doteq \varpi^2(x, y)g_{ab}(x)y^ay^b$  and use

$${}^\varpi g_{ab} \doteq \frac{1}{2} \frac{\partial^2 {}^\varpi\mathcal{L}}{\partial y^a \partial y^b} \quad (71)$$

as a Lagrange Hessian for (47). A space  $GL^n = (M, \varpi g_{ij}(x, y))$  possess a weakly regular conformally deformed metric if  $L^n = [M, L = \sqrt{|\varpi \mathcal{L}|}]$  is a Lagrange space. We can construct a canonical N-connection  $\varpi N_i^a$  following formulas (51), using  $\varpi \mathcal{L}$  instead of  $\mathcal{L}$  and  $\varpi g_{ab}$  instead of  $\mathcal{L} g_{ab}$  (50), and define a d-metric on  $TM$ ,

$$\varpi \mathbf{g} = \varpi g_{ij}(x, y) dx^i \otimes dx^j + \varpi g_{ij}(x, y) \varpi \mathbf{e}^i \otimes \varpi \mathbf{e}^j, \quad (72)$$

where  $\varpi \mathbf{e}^i = dy^i + \varpi N_i^j dx^j$ . The canonical d-connection and corresponding curvatures are constructed as in generalized Lagrange geometry but using  $\varpi \mathbf{g}$ .

For the d-metric (72), the model is elaborated for tangent bundles with holonomic vertical frame structure. The linear operator  $\mathbf{F}$  defining the almost complex structure acts on  $\mathbf{TM}$  following formulas  $\mathbf{F}(\varpi \mathbf{e}_i) = -\partial_i$  and  $\mathbf{F}(\partial_i) = \varpi \mathbf{e}_i$ , when  $\mathbf{F} \circ \mathbf{F} = -\mathbf{I}$ , for  $\mathbf{I}$  being the unity matrix. The operator  $\mathbf{F}$  reduces to a complex structure if and only if the h-distribution is integrable.

The metric  $\varpi \mathbf{g}$  (72) induces a 2-form associated to  $\mathbf{F}$  following formulas  $\varpi \theta(\mathbf{X}, \mathbf{Y}) \doteq \varpi \mathbf{g}(\mathbf{F}\mathbf{X}, \mathbf{Y})$  for any d-vectors  $\mathbf{X}$  and  $\mathbf{Y}$ . In local form, we have  $\varpi \theta = \varpi g_{ij}(x, y) dy^i \wedge dx^j$ . The canonical d-connection  $\varpi \hat{\mathbf{D}}$ , with N-adapted coefficients  $\varpi \Gamma_{\beta\gamma}^\alpha = \begin{pmatrix} \varpi \hat{L}^i_{jk}, & \varpi \hat{C}^a_{bc} \end{pmatrix}$ , and corresponding d-curvature has to be computed with  $\check{e}_b^{\check{b}} = \delta_b^{\check{b}}$  and  $\varpi g_{ij}$  used instead of  $g_{ij}$ .

The model of almost Hermitian gravity  $H^{2n}(\mathbf{TM}, \varpi \mathbf{g}, \mathbf{F})$  can be applied in order to construct different extensions of general relativity to geometric quantum models on tangent bundle [70]. Such models will result positively in violation of local Lorentz symmetry, because the geometric objects depend on fiber variables  $y^a$ . The quasi-classical corrections can be obtained in the approximation  $\varpi \sim 1$ . We omit in this work consideration of quantum models, but note that Finsler methods and almost Kähler geometry seem to be very useful for such generalizations of Einstein gravity.

## 5 Finsler–Lagrange Metrics in Einstein & String Gravity

We consider certain general conditions when Lagrange and Finsler structures can be modelled as exact solutions in string and Einstein gravity. Then, we analyze two explicit examples of exact solutions of the Einstein equations modelling generalized Lagrange–Finsler geometries and nonholonomic deformations of physically valuable equations in Einstein gravity to such locally anisotropic configurations.



## 5.1 Einstein spaces modelling generalized Finsler structures

In this section, we outline the calculation leading from generalized Lagrange and Finsler structures to exact solutions in gravity. Let us consider

$${}^\varepsilon \check{\mathbf{g}} = {}^\varepsilon g_{i'j'}(x^{k'}, y^{l'}) \left( e^{i'} \otimes e^{j'} + \check{\mathbf{e}}^{i'} \otimes \check{\mathbf{e}}^{j'} \right), \quad (73)$$

where  ${}^\varepsilon g_{i'j'}$  can be any metric defined by nonholonomic transforms (52) or a v-metric  ${}^{\mathcal{L}}g_{ij}$  (50),  ${}^Lg_{ij}$  (45), or  ${}^Fg_{ij}$  (48). The co-frame h- and v-bases

$$e^{i'} = e^{i'}_i(x, y) dx^i, \quad \check{\mathbf{e}}^{a'} = \check{e}^{a'}_a(x, y) \delta y^a = \check{e}^{a'}_a (dy^a + {}_iN_i^a dx^i) = e^{a'} + {}_i\check{N}_{i'}^a e^{i'},$$

define  $e^{a'} = \check{e}^{a'}_a dy^a$  and  ${}_i\check{N}_{i'}^a = \check{e}^{a'}_a {}_iN_i^a e^{i'}$ , when

$${}^\varepsilon g_{i'j'} = e^{i'}_i e^{j'}_{i'} {}_i g_{ij}, \quad {}_i h_{ab} = {}^\varepsilon g_{a'b'} \check{e}^{b'}_a \check{e}^{b'}_b, \quad {}_i N_i^a = \eta_i^a(x, y) {}^\varepsilon N_i^a, \quad (74)$$

where we do not consider summation on indices for "polarization" functions  $\eta_i^a$  and  ${}^\varepsilon N_i^a$  is a canonical connection corresponding to  ${}^\varepsilon g_{i'j'}$ .

The d-metric (73) is equivalently transformed into the d-metric

$$\begin{aligned} {}^\varepsilon \check{\mathbf{g}} &= {}_i g_{ij}(x) dx^i \otimes dx^j + {}_i h_{ab}(x, y) \delta y^a \otimes \delta y^b, \\ \delta y^a &= dy^a + {}_i N_i^a(x, y) dx^i, \end{aligned} \quad (75)$$

where the coefficients  ${}_i g_{ij}(x)$ ,  ${}_i h_{ab}(x, y)$  and  ${}_i N_i^a(x, y)$  are constrained to be defined by a class of exact solutions (66), in string gravity, or (69), in Einstein gravity. If it is possible to get the limit  $\eta_i^a \rightarrow 1$ , we can say that an exact solution (75) models exactly a respective (generalized) Lagrange, or Finsler, configuration. We argue that we define a nonholonomic deformation of a Finsler (Lagrange) space given by data  ${}^\varepsilon g_{i'j'}$  and  ${}^\varepsilon N_i^a$  as a class of exact solutions of the Einstein equations given by data  ${}_i g_{ij}$ ,  ${}_i h_{ab}$  and  ${}_i N_i^a$ , for any  $\eta_i^a(x, y) \neq 1$ . Such constructions are possible, if certain nontrivial values of  $e^{i'}_i, \check{e}^{a'}_a$  and  $\eta_i^a$  can be algebraically defined from relations (74) for any defined sets of coefficients of the d-metric (73) and (75).

Expressing a solution in the form (73), we can define the corresponding almost Hermitian 1-form  $\check{\theta} = g_{i'j'}(x, y) \check{e}^{j'} \wedge e^{i'}$ , and construct an almost Hermitian geometry characterizing this solution for  $\check{\mathbf{F}}(\mathbf{e}_{i'}) = -\check{e}_{i'}$  and  $\check{\mathbf{F}}(\check{\mathbf{e}}_{i'}) = \mathbf{e}_{i'}$ , when  $\mathbf{e}_{i'} = e^{i'}_i \left( \frac{\partial}{\partial x^i} - {}_i N_i^a \frac{\partial}{\partial y^a} \right) = e_{i'} - {}_i \check{N}_{i'}^a \check{e}_{a'}$ . This is convenient for further applications to certain models of quantum gravity and geometry. For explicit constructions of the solutions, it is more convenient to work with parametrizations of type (75).

Finally, in this section, we note that the general properties of integral varieties of such classes of solutions are discussed in Refs. [66, 77].

## 5.2 Deformation of Einstein exact solutions into Lagrange–Finsler metrics

Let us consider a metric ansatz  ${}_1g_{\alpha\beta}$  (16) with quadratic metric interval

$$\begin{aligned} ds^2 = & {}_1g_1(x^1, x^2) (dx^1)^2 + {}_1g_2(x^1, x^2) (dx^2)^2 \\ & + {}_1h_3(x^1, x^2, v) [dv + {}_1w_1(x^1, x^2, v)dx^1 + {}_1w_2(x^1, x^2, v)dx^2]^2 \\ & + {}_1h_4(x^1, x^2, v) [dy^4 + {}_1n_1(x^1, x^2, v)dx^1 + {}_1n_2(x^1, x^2, v)dx^2]^2 \end{aligned} \quad (76)$$

defining an exact solution of the Einstein equations (7), for the Levi–Civita connection, when the source  $\Upsilon_{\alpha\beta}$  is zero or defined by a cosmological constant. We parametrize the coordinates in the form  $u^\alpha = (x^1, x^2, y^3 = v, y^4)$  and the N-connection coefficients as  ${}_1N_i^3 = {}_1w_i$  and  ${}_1N_i^4 = {}_1n_i$ .

We nonholonomically deform the coefficients of the **primary** d-metric (76), similarly to (74), when the **target** quadratic interval

$$\begin{aligned} ds_\eta^2 = & g_i (dx^i)^2 + h_a (dy^a + N_i^a dx^i)^2 = e^{i'}_i e^{j'}_j {}^\varepsilon g_{i'j'} dx^{i'} dx^{j'} \\ & + \check{e}^{a'}_a \check{e}^{b'}_b {}^\varepsilon g_{a'b'} (dy^a + \eta_i^a N_i^a dx^i) (dy^b + \eta_j^b N_j^b dx^j) \end{aligned} \quad (77)$$

can be equivalently parametrized in the form

$$\begin{aligned} ds_\eta^2 = & \eta_j {}_1g_j(x^i) (dx^j)^2 \\ & + \eta_3(x^i, v) {}_1h_3(x^i, v) [dv + {}^w\eta_i(x^k, v) {}^\varepsilon w_i(x^k, v) dx^i]^2 \\ & + \eta_4(x^i, v) {}_1h_4(x^i, v) [dy^4 + {}^n\eta_i(x^k, v) {}^\varepsilon n_i(x^k, v) dx^i]^2, \end{aligned} \quad (78)$$

similarly to ansatz (53), and defines a solution of type (66) (with N-connection coefficients (67) and (68)), for the canonical d-connection, or a solution of type (69) with the coefficients subjected to solve the conditions (70).

The class of target metrics (77) and (78) defining the result of a nonholonomic deformation of the primary data  $[{}_1g_i, {}_1h_a, {}_1N_i^b]$  to a Finsler–Lagrange configuration  $[{}^\varepsilon g_{i'j'}, {}^\varepsilon N_j^b]$  are parametrized by vales  $e^{i'}_i, \check{e}^{a'}_a$  and  $\eta_i^a$ . These values can be expressed in terms of some generation and integration functions and the coefficients of the primary and Finsler like d-metrics and N-connections in such a manner when a primary class of exact solutions is transformed into a "more general" class of exact solutions. In a particular case, we can search for solutions when the target metrics transform into primary metrics under some infinitesimal limits.

In general form, the solutions of equations (40) transformed into the system of partial differential equations (57)–(60), for the d-metrics (77),

equivalently (78), are given by corresponding sets of frame coefficients

$$\begin{aligned} e_1^{1'} &= \sqrt{|\eta_1|} \sqrt{|g_1|} \times {}^\varepsilon E_+, \quad e_1^{2'} = \sqrt{|\eta_2|} \sqrt{|g_1|} / {}^\varepsilon E_+, \\ e_2^{1'} &= -\sqrt{|\eta_2|} \sqrt{|g_2|} \times g_{1'2'} / {}^\varepsilon E_-, \quad e_2^{2'} = \sqrt{|\eta_2|} \sqrt{|g_2|} \times {}^\varepsilon E_-, \end{aligned} \quad (79)$$

$$\begin{aligned} e_3^{3'} &= \sqrt{|\eta_3|} \sqrt{|h_3|} \times {}^\varepsilon E_+, \quad e_3^{4'} = \sqrt{|\eta_3|} \sqrt{|h_3|} / {}^\varepsilon E_+, \\ e_4^{3'} &= -\sqrt{|\eta_4|} \sqrt{|h_4|} \times g_{1'2'} / {}^\varepsilon E_-, \quad e_4^{4'} = \sqrt{|\eta_4|} \sqrt{|h_4|} \times {}^\varepsilon E_-, \end{aligned} \quad (80)$$

where  ${}^\varepsilon E_\pm = \sqrt{|{}^\varepsilon g_{1'1'} {}^\varepsilon g_{2'2'} [({}^\varepsilon g_{1'1'})^2 {}^\varepsilon g_{2'2'} \pm ({}^\varepsilon g_{1'2'})^3]^{-1}|}$  and h-polarizations  $\eta_j$  are defined from  $g_j = \eta_j {}_1g_j(x^i) = \epsilon_j e^{\psi(x^i)}$ , with signatures  $\epsilon_i = \pm 1$ , for  $\psi(x^i)$  being a solution of the 2D equation

$$\epsilon_1 \psi^{\bullet\bullet} + \epsilon_2 \psi'' = \lambda, \quad (81)$$

for a given source  $\Upsilon_4(x^i) = \lambda$ , and the v-polarizations  $\eta_a$  defined from the data  $h_a = \eta_a {}_1h_a$ , for

$$\begin{aligned} h_3 &= \epsilon_3 h_0^2(x^i) [f^*(x^i, v)]^2 |{}^\lambda \varsigma(x^i, v)|, \quad h_4 = \epsilon_4 [f(x^i, v) - f_0(x^i)]^2, \\ {}^\lambda \varsigma &= \varsigma_{[0]}(x^i) - \frac{\epsilon_3}{8} \lambda h_0^2(x^i) \int f^*(x^i, v) [f(x^i, v) - f_0(x^i)] dv, \end{aligned} \quad (82)$$

for  $\Upsilon_2(x^k, v) = \lambda$ . The polarizations  $\eta_i^a$  of N-connection coefficients  $N_i^3 = w_i = {}^w\eta_i(x^k, v) {}^\varepsilon w_i(x^k, v)$ ,  $N_i^4 = n_i = {}^n\eta_i(x^k, v) {}^\varepsilon n_i(x^k, v)$  are computed from respective formulas

$${}^w\eta_i {}^\varepsilon w_i = -\frac{\partial_i {}^\lambda \varsigma(x^k, v)}{{}^\lambda \varsigma^*(x^k, v)}, \quad (83)$$

$${}^n\eta_k {}^\varepsilon n_k = {}^1n_k(x^i) + {}^2n_k(x^i) \int \frac{[f^*(x^i, v)]^2 {}^\lambda \varsigma(x^i, v)}{[f(x^i, v) - f_0(x^i)]^3} dv. \quad (84)$$

We generate a class of exact solutions for Einstein spaces with  $\Upsilon_2 = \Upsilon_4 = \lambda$  if the integral varieties defined by  $g_j, h_a, w_i$  and  $n_i$  are subjected to constraints (70).

### 5.3 Solitonic pp-waves and their effective Lagrange spaces

Let us consider a d-metric of type (76),

$$\delta s_{[pw]}^2 = -dx^2 - dy^2 - 2\kappa(x, y, v) dv^2 + dp^2 / 8\kappa(x, y, v), \quad (85)$$

where the local coordinates are  $x^1 = x$ ,  $x^2 = y$ ,  $y^3 = v$ ,  $y^4 = p$ , and the nontrivial metric coefficients are parametrized  ${}_1g_1 = -1$ ,  ${}_1g_2 = -1$ ,  ${}_1h_3 = -2\kappa(x, y, v)$ ,  ${}_1h_4 = 1/8 \kappa(x, y, v)$ . This is vacuum solution of the Einstein equation defining pp-waves [47]: for any  $\kappa(x, y, v)$  solving  $\kappa_{xx} + \kappa_{yy} = 0$ , with  $v = z + t$  and  $p = z - t$ , where  $(x, y, z)$  are usual Cartesian coordinates and

$t$  is the time like coordinate. Two explicit examples of such solutions are given by  $\kappa = (x^2 - y^2) \sin v$ , defining a plane monochromatic wave, or by  $\kappa = xy/(x^2 + y^2)^2 \exp[v_0^2 - v^2]$ , for  $|v| < v_0$ , and  $\kappa = 0$ , for  $|v| \geq v_0$ , defining a wave packet travelling with unit velocity in the negative  $z$  direction.

We nonholonomically deform the vacuum solution (85) to a d-metric of type (78)

$$\begin{aligned} ds_\eta^2 = & -e^{\psi(x,y)} \left[ (dx)^2 + (dy)^2 \right] \\ & - \eta_3(x, y, v) \cdot 2\kappa(x, y, v) \left[ dv + {}^w\eta_i(x, y, v, p) {}^\varepsilon w_i(x, y, v, p) dx^i \right]^2 \\ & + \eta_4(x, y, v) \cdot \frac{1}{8\kappa(x, y, v)} \left[ dy^4 + {}^n\eta_i(x, y, v, p) {}^\varepsilon n_i(x, y, v, p) dx^i \right]^2, \end{aligned} \quad (86)$$

where the polarization functions  $\eta_1 = \eta_2 = e^{\psi(x,y)}$ ,  $\eta_{3,4}(x, y, v)$ ,  ${}^w\eta_i(x, y, v)$  and  ${}^n\eta_i(x, y, v)$  have to be defined as solutions in the form (81), (82), (83) and (84) for a string gravity ansatz (65),  $\lambda = \lambda_H^2/2$ , and a prescribed (in this section) analogous mechanical system with

$$N_i^a = \{w_i(x, y, v) = {}^w\eta_i {}^L w_i, n_i(x, y, v) = {}^n\eta_i {}^\varepsilon n_i\} \quad (87)$$

for  $\varepsilon = L(x, y, v, p)$  considered as regular Lagrangian modelled on a N-anholonomic manifold with holonomic coordinates  $(x, y)$  and nonholonomic coordinates  $(v, p)$ .

A class of 3D solitonic configurations can be defined by taking a polarization function  $\eta_4(x, y, v) = \eta(x, y, v)$  as a solution of solitonic equation<sup>11</sup>

$$\eta^{\bullet\bullet} + \epsilon(\eta' + 6\eta \eta^* + \eta^{***})^* = 0, \quad \epsilon = \pm 1, \quad (88)$$

and  $\eta_1 = \eta_2 = e^{\psi(x,y)}$  as a solution of (81) written as

$$\psi^{\bullet\bullet} + \psi'' = \frac{\lambda_H^2}{2}. \quad (89)$$

Introducing the above stated data for the ansatz (86) into the equation (58), we get two equations relating  $h_3 = \eta_3 {}^L h_3$  and  $h_4 = \eta_4 {}^L h_4$ ,

$$\eta_4 = 8 \kappa(x, y, v) \left[ h_4^{[0]}(x, y) + \frac{1}{2\lambda_H^2} e^{2\eta(x,y,v)} \right] \quad (90)$$

$$|\eta_3| = \frac{e^{-2\eta(x,y,v)}}{2\kappa^2(x, y, v)} \left[ \left( \sqrt{|\eta_4(x, y, v)|} \right)^* \right]^2, \quad (91)$$

where  $h_4^{[0]}(x, y)$  is an integration function.

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<sup>11</sup>as a matter of principle we can consider that  $\eta$  is a solution of any 3D solitonic, or other, nonlinear wave equation.

Having defined the coefficients  $h_a$ , we can solve the equations (59) and (60) expressing the coefficients (61) and (62) through  $\eta_3$  and  $\eta_4$  defined by pp- and solitonic waves as in (91) and (90). The corresponding solutions

$$w_1 = {}^w\eta_1 {}^Lw_1 = (\phi^*)^{-1} \partial_x \phi, \quad w_2 = {}^w\eta_1 {}^Lw_1 = (\phi^*)^{-1} \partial_y \phi, \quad (92)$$

$$n_i = n_i^{[0]}(x, y) + n_i^{[1]}(x, y) \int \left| \eta_3(x, y, v) \eta_4^{-3/2}(x, y, v) \right| dv, \quad (93)$$

are for  $\phi^* = \partial\phi/\partial v$ , see formulas (62), where  $n_i^{[0]}(x, y)$  and  $n_i^{[1]}(x, y)$  are integration functions. The values  $e^{\psi(x, y)}$ ,  $\eta_3$  (91),  $\eta_4$  (90),  $w_i$  (92) and  $n_i$  (93) for the ansatz (86) completely define a nonlinear superpositions of solitonic and pp-waves as an exact solution of the Einstein equations in string gravity if there are prescribed some initial values for the nonlinear waves under consideration. In general, such solutions depend on some classes of generation and integration functions.

It is possible to give a regular Lagrange analogous interpretation of an explicit exact solution (86) if we prescribe a regular Lagrangian  $\varepsilon = L(x, y, v, p)$ , with Hessian  ${}^Lg_{i'j'} = \frac{1}{2} \frac{\partial^2 L}{\partial y^{i'} \partial y^{j'}}$ , for  $x^{i'} = (x, y)$  and  $y^{a'} = (v, p)$ . Introducing the values  ${}^Lg_{i'j'}$ ,  $\eta_1 = \eta_2 = e^{\psi}$ ,  $\eta_3, \eta_4$  and  ${}_1h_3, {}_1h_4$ , all defined above, into (79) and (80), we compute the vierbein coefficients  $e^{i'}_i$  and  $\check{e}^{a'}_a$  which allows us to redefine equivalently the quadratic element in the form (77), for which the N-connection coefficients  ${}^LN_i^a$  (46) are nonholonomically deformed to  $N_i^a$  (87). With respect to such nonholonomic frames of references, an observer "swimming in a string gravitational ocean of interacting solitonic and pp-waves" will see his world as an analogous mechanical model defined by a regular Lagrangian  $L$ .

#### 5.4 Finsler-solitonic pp-waves in Schwarzschild spaces

We consider a primary quadratic element

$$\delta s_0^2 = -d\xi^2 - r^2(\xi) d\vartheta^2 - r^2(\xi) \sin^2 \vartheta d\varphi^2 + \varpi^2(\xi) dt^2, \quad (94)$$

where the nontrivial metric coefficients are

$${}_1g_2 = -r^2(\xi), \quad {}_1h_3 = -r^2(\xi) \sin^2 \vartheta, \quad {}_1h_4 = \varpi^2(\xi), \quad (95)$$

with  $x^1 = \xi, x^2 = \vartheta, y^3 = \varphi, y^4 = t, {}_1g_1 = -1, \xi = \int dr \left| 1 - \frac{2\mu}{r} \right|^{1/2}$  and  $\varpi^2(r) = 1 - \frac{2\mu}{r}$ . For  $\mu$  being a point mass, the element (94) defines the Schwarzschild solution written in spacetime spherical coordinates  $(r, \vartheta, \varphi, t)$ .

Our aim, is to find a nonholonomic deformation of metric (94) to a class of new vacuum solutions modelled by certain types of Finsler geometries.

The target stationary metrics are parametrized in the form similar to (78), see also (69),

$$\begin{aligned} ds_\eta^2 &= -e^{\psi(\xi, \vartheta)} \left[ (d\xi)^2 + r^2(\xi)(d\vartheta)^2 \right] \\ &- \eta_3(\xi, \vartheta, \varphi) \cdot r^2(\xi) \sin^2 \vartheta [d\varphi + {}^w\eta_i(\xi, \vartheta, \varphi, t) {}^Fw_i(\xi, \vartheta, \varphi, t) dx^i]^2 \\ &+ \eta_4(\xi, \vartheta, \varphi) \cdot \varpi^2(\xi) [dt + {}^n\eta_i(\xi, \vartheta, \varphi, t) {}^Fn_i(\xi, \vartheta, \varphi, t) dx^i]^2. \end{aligned} \quad (96)$$

The polarization functions  $\eta_1 = \eta_2 = e^{\psi(\xi, \vartheta)}$ ,  $\eta_a(\xi, \vartheta, \varphi)$ ,  ${}^w\eta_i(\xi, \vartheta, \varphi)$  and  ${}^n\eta_i(\xi, \vartheta, \varphi)$  have to be defined as solutions of (70) for  $\Upsilon_2 = \Upsilon_4 = 0$  and a prescribed (in this section) locally anisotropic, on  $\varphi$ , geometry with  $N_i^a = \{w_i(\xi, \vartheta, \varphi) = {}^w\eta_i {}^Fw_i, n_i(\xi, \vartheta, \varphi) = {}^n\eta_i {}^Fn_i\}$ , for  $\varepsilon = F^2(\xi, \vartheta, \varphi, t)$  considered as a fundamental Finsler function for a Finsler geometry modelled on a N-anholonomic manifold with holonomic coordinates  $(r, \vartheta)$  and nonholonomic coordinates  $(\varphi, t)$ . We note that even the values  ${}^w\eta_i$ ,  ${}^Fn_i$ , and  ${}^Fn_i$  can depend on time like variable  $t$ , such dependencies must result in N-connection coefficients of type  $N_i^a(\xi, \vartheta, \varphi)$ .

Putting together the coefficients solving the Einstein equations (58)–(60) and (70), the class of vacuum solutions in general relativity related to (96) can be parametrized in the form

$$\begin{aligned} ds_\eta^2 &= -e^{\psi(\xi, \vartheta)} \left[ (d\xi)^2 + r^2(\xi)(d\vartheta)^2 \right] \\ &- h_0^2 [b^*(\xi, \vartheta, \varphi)]^2 [d\varphi + w_1(\xi, \vartheta)d\xi + w_2(\xi, \vartheta)d\vartheta]^2 \\ &+ [b(\xi, \vartheta, \varphi) - b_0(\xi, \vartheta)]^2 [dt + n_1(\xi, \vartheta)d\xi + n_2(\xi, \vartheta)d\vartheta]^2, \end{aligned} \quad (97)$$

where  $h_0 = \text{const}$  and the coefficients are constrained to solve the equations

$$\psi^{\bullet\bullet} + \psi'' = 0, \quad w_1' - w_2^\bullet + w_2w_1^* - w_1w_2^* = 0, \quad n_1' - n_2^\bullet = 0, \quad (98)$$

for instance, for  $w_1 = (b^*)^{-1}(b + b_0)^\bullet$ ,  $w_2 = (b^*)^{-1}(b + b_0)'$ ,  $n_2^\bullet = \partial n_2 / \partial \xi$  and  $n_1' = \partial n_1 / \partial \vartheta$ .

The polarization functions relating (97) to (96), are computed

$$\begin{aligned} \eta_1 &= \eta_2 = e^{\psi(\xi, \vartheta)}, \quad \eta_3 = [h_0 b^* / r(\xi) \sin \vartheta]^2, \quad \eta_4 = [(b - b_0) / \varpi]^2, \\ {}^w\eta_i &= w_i(\xi, \vartheta) / {}^Fw_i(\xi, \vartheta, \varphi, t), \quad {}^n\eta_i = n_i(\xi, \vartheta) / {}^Fn_i(\xi, \vartheta, \varphi, t). \end{aligned} \quad (99)$$

The next step is to chose a Finsler geometry which will model (97), equivalently (96), as a Finsler like d-metric (77). For a fundamental Finsler function  $F = F(\xi, \vartheta, \varphi, t)$ , where  $x^{i'} = (\xi, \vartheta)$  are h-coordinates and  $y^{a'} = (\varphi, t)$  are v-coordinates, we compute  ${}^Fg_{a'b'} = (1/2)\partial^2 F / \partial y^{a'} \partial y^{b'}$  following formulas (48) and parametrize the Cartan N-connection as  ${}^CN_i^a = \{ {}^Fw_i, {}^Fn_i \}$ . Introducing the values (95),  ${}^Fg_{i'j'}$  and polarization functions (99) into (79) and (80), we compute the vierbein coefficients  $e^{i'}_i$  and  $\tilde{e}^{a'}_a$  which allows us to redefine equivalently the quadratic element in the form (77), in this case, for

a Finsler configuration for which the N-connection coefficients  ${}^C N_i^a$  (46) are nonholonomically deformed to  $N_i^a$  satisfying the last two conditions in (98). With respect to such nonholonomic frames of references, an observer "swimming in a locally anisotropic gravitational ocean" will see the nonholonomically deformed Schwarzschild geometry as an analogous Finsler model defined by a fundamental Finsler function  $F$ .

## 6 Outlook and Conclusions

In this review article, we gave a self-contained account of the core developments on generalized Finsler–Lagrange geometries and their modelling on (pseudo) Riemannian and Riemann–Cartan manifolds provided with preferred nonholonomic frame structure. We have shown how the Einstein gravity and certain string models of gravity with torsion can be equivalently reformulated in the language of generalized Finsler and almost Hermitian/Kähler geometries. It was also argued that former criticism and conclusions on experimental constraints and theoretical difficulties of Finsler like gravity theories were grounded only for certain classes of theories with metric noncompatible connections on tangent bundles and/or resulting in violation of local Lorentz symmetry. We emphasized that there were omitted the results when for some well defined classes of nonholonomic transforms of geometric structures we can model geometric structures with local anisotropy, of Finsler–Lagrange type, and generalizations, on (pseudo) Riemann spaces and Einstein manifolds.

Our idea was to consider not only some convenient coordinate and frame transforms, which simplify the procedure of constructing exact solutions, but also to define alternatively new classes of connections which can be employed to generate new solutions in gravity. We proved that the solutions for the so-called canonical distinguished connections can be equivalently re-defined for the Levi Civita connection and/or constrained to define integral varieties of solutions in general relativity.

The main conclusion of this work is that we can avoid all existing experimental restrictions and theoretical difficulties of Finsler physical models if we work with metric compatible Finsler like structures on nonholonomic (Riemann, or Riemann–Cartan) manifolds but not on tangent bundles. In such cases, all nonholonomic constructions modelled as exact solutions of the Einstein and matter field equations (with various string, quantum field ... corrections) are compatible with the standard paradigm in modern physics.

In other turn, we emphasize that in quantum gravity, statistical and thermodynamical models with local anisotropy, gauge theories with constraints and broken symmetry and in geometric mechanics, nonholonomic configurations on (co) tangent bundles, of Finsler type and generalizations, metric compatible or with nonmetricity, seem to be also very important.

Various directions in generalized Finsler geometry and applications has matured enough so that some tenths of monographs have been written, including some recent and updated: we cite here [33, 37, 38, 34, 35, 39, 36, 10, 77, 62, 76, 8, 54, 2, 4, 5, 6, 7, 14]. These monographs approach and present the subjects from different perspectives depending, of course, on the authors own taste, historical period and interests both in geometry and physics. The monograph [77] summarizes and develops the results oriented to application of Finsler methods to standard theories of gravity (on nonholonomic manifolds, not only on tangent bundles) and their noncommutative generalizations; it was also provided a critical analysis of the constructions with nonmetricity and violations of local Lorentz symmetry.

Finally, we suggest the reader to see a more complete review [72] discussing applications of Finsler and Lagrange geometry both to standard and nonstandard models of physics (presenting a variant which was not possible to be published because of limit of space in this journal).

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